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ON CERTAIN INVESTIGATION OF PRODUCT OF SPECIAL FUNCTION USING FRACTIONAL CALCULUS

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ABSTRACT

The paper is devoted to study the generalized fractional calculus of arbitrary complex order for the $\overline{\mathrm{H}}$-function defined by Inayat Hussain [8]. The classical fractional integrals and derivatives of Riemann-Liouville type are treated. The considered generalized fractional integration and differentiation operators contain the Gauss Hypergeometric function as a kernel and generalize classical Riemann-Liouville, Erdelyi-Kober types ones. It is proved that the generalized fractional integrals and derivatives of $\overline{\mathrm{H}}$-function turn also out $\overline{\mathrm{H}}$-functions but of greater order. Especially, the obtained results define more precise and general ones than known. Corresponding assertion for Riemann-Liouville and Erdelyi-Kober fractional integral operators are also presented.

Keywords: Mellin Barnes Contour Intrgrals, Fractional Operators

I. INTRODUCTION

By evaluating certain Feynman integrals which arise naturally in perturbation calculations of the equilibrium properties of a magnetic model of phase transitions, Inayat-Hussain [7] derived a number of interesting properties and characteristics of hypergeometric functions of one and more variables. While presenting some further examples of the use of these Feynman integrals, Inayat-Hussain [8] was led to a novel generalization of the familiar H-function of Charles Fox [4]. This function is popularly known as $\overline{H}$-function and contains some new special cases, such as the polylogarithm of a complex order and the exact partition function of the Gaussian model in Statistical mechanics. In terms of Mellin-Barnes contour integral, it is defined as follows:

$$\overline{H}_{P,Q}^{M,N}\left(z\left|\begin{matrix}(a_j,\alpha_j;A_j)_{1,N},(a_j,\alpha_j)_{N+1,P}\\(b_j,\beta_j)_{1,M},(b_j,\beta_j;B_j)_{M+1,Q}\end{matrix}\right.\right)=\frac{1}{2\pi i}\int_L\overline{\phi}(\xi)z^{\xi}d\xi,\qquad (z\neq 0)\tag{1.1}$$

where $$\overline{\phi}(\xi)=\frac{\prod_{j=1}^{M}\Gamma(b_j-\beta_j\xi)\prod_{j=1}^{N}\{\Gamma(1-a_j+\alpha_j\xi)\}^{A_j}}{\prod_{j=M+1}^{Q}\{\Gamma(1-b_j+\beta_j\xi)\}^{B_j}\prod_{j=N+1}^{P}\Gamma(a_j-\alpha_j\xi)}.\tag{1.2}$$

Here $a_j\,(j=1,...,p)$ and $b_j\,(j=1,...,q)$ are complex parameters, $A_j>0\,(j=1,...,p)$ and $B_j>0\,(j=1,...,q)$ and the exponents $\alpha_j\,(j=1,...,p)$ and $\beta_j\,(j=1,...,q)$ can take noninteger values. The following sufficient conditions for the absolute convergence of the defining integral for $\overline{H}$ - Function given by (1.1) have been recently given by Gupta, Jain and Agrawal [5]:

(i) $|\arg(z)|<1/2\,\Omega\pi\ and\ \Omega>0$,

(ii) $|\arg(z)|=1/2\,\Omega\pi\ and\ \Omega\geq 0$,

and (a) $\mu\neq 0$ and the contour L is so chosen that $(c\mu+\lambda+1)$ <0,

(b) $\mu=0\ and\ (\lambda+1)<0$, (1.3)

where

$$\Omega=\sum_{1}^{M}\beta_j+\sum_{1}^{N}\alpha_jA_j-\sum_{M+1}^{Q}\beta_jB_j-\sum_{N+1}^{P}\alpha_j\,,\tag{1.4}$$

$$\mu=\sum_{1}^{N}\alpha_jA_j+\sum_{N+1}^{P}\alpha_j-\sum_{1}^{M}\beta_j-\sum_{M+1}^{Q}\beta_jB_j\,,\tag{1.5}$$

$$\lambda=\mathrm{Re}\left(\sum_{1}^{M}b_j+\sum_{M+1}^{Q}b_jB_j-\sum_{1}^{N}a_jA_j-\sum_{N+1}^{P}a_j\right).\ +\frac{1}{2}\left(-M-\sum_{M+1}^{Q}B_j+\sum_{1}^{N}A_j+P-N\right)\tag{1.6}$$

It may be noted that the conditions of validity given above are more general than those given earlier [1].

The following series representation of the $\overline{H}$ -function was given by Rathie [12].

$$\overline{H}_{P,Q}^{M,N}\left(z\left|\begin{matrix}(a_j,\alpha_j;A_j)_{1,N},(a_j,\alpha_j)_{N+1,P}\\(b_j,\beta_j)_{1,M},(b_j,\beta_j;B_j)_{M+1,Q}\end{matrix}\right.\right)=\sum_{\upsilon=1}^{M}\sum_{P=0}^{\infty}\overline{\theta}(S_{P,\upsilon})z^{S_{P,\upsilon}}\tag{1.7}$$

where $$\overline{\theta}(S_{P,\upsilon})=\frac{\prod_{j=1,j\neq\upsilon}^{M}\Gamma(b_j-\beta_jS_{P,\upsilon})\prod_{j=1}^{N}\{\Gamma(1-a_j+\alpha_jS_{P,\upsilon})\}^{A_j}(-1)^P}{\prod_{j=M+1}^{Q}\{\Gamma(1-b_j+\beta_jS_{P,\upsilon})\}^{B_j}\prod_{j=N+1}^{P}\Gamma(a_j-\alpha_jS_{P,\upsilon})P!\,\beta_\upsilon}$$

The following behavior of $\overline{H}_{P,Q}^{M,N}[z]$ for small and large value of z as recorded by Saxena [16, p.112, eqs. (2.3) and (2.4)] will be required in the sequel

$$\overline{H}_{P,Q}^{M,N}[z] = O\left[|z|^{\alpha}\right] \text{for small z, where } \alpha = \min_{1\le j\le M}\left[\operatorname{Re}\left(\frac{b_j}{\beta_j}\right)\right] \tag{1.9}$$

$$\overline{H}_{P,Q}^{M,N}[z] = O\left[|z|^{\beta}\right] \text{for large z, where } \beta = \max_{1\le j\le N}\left[\operatorname{Re}\left(\frac{a_j-1}{\alpha_j}\right)\right] \tag{1.10}$$

and the conditions (1.3) are satisfied.

Also $S_V^U[x]$ occurring in the sequel denotes the general class of polynomials [17, p.1, Eq. (1)]

$$S_V^U[x] = \sum_{R=0}^{[V/U]} (-V)_{UR} A(V,R)\frac{x^R}{R!}, \tag{1.11}$$

where U is an arbitrary positive integer, V=0,1,2,… and the coefficients A(V,R) $A_{V,R}\,(V,R\ge 0)$ are arbitrary constants, real or complex. On suitably specializing the coefficients $A_{V,R}$ yields a number of known polynomials as its special cases.

II CLASSICAL AND GENERALIZED FRACTIONAL CALCULUS OPERATORS

For α ∈ C (Re (α) > 0), the Riemann-Liouville left and right-sided fractional calculus operators are defined as the following ([15, sections 2.3 and 2.4)]:

$$(I_{0+}^{\alpha} f)(x) = \frac{1}{\Gamma(\alpha)}\int_0^x \frac{f(t)}{(x-t)^{1-\alpha}}\,dt, \quad (x>0) \tag{2.1}$$

$$(I_{0-}^{\alpha} f)(x) = \frac{1}{\Gamma(\alpha)}\int_x^\infty \frac{f(t)}{(t-x)^{1-\alpha}}\,dt, \quad (x>0) \tag{2.2}$$

and

$$(D_{0+}^{\alpha} f)(x) = \left(\frac{d}{dx}\right)^{[\operatorname{Re}(\alpha)+1]}\left(I_{0+}^{1-\alpha+[\operatorname{Re}(\alpha)]} f\right)(x)$$
$$= \left(\frac{d}{dx}\right)^{[\operatorname{Re}(\alpha)]+1}\frac{1}{\Gamma(1-\alpha+[\operatorname{Re}(\alpha)])}\int_0^x \frac{f(t)}{(x-t)^{\alpha-[\operatorname{Re}(\alpha)]}}\,dt, \quad (x>0) \tag{2.3}$$

$$(D_{-}^{\alpha} f)(x) = \left(-\frac{d}{dx}\right)^{[\operatorname{Re}(\alpha)]+1}\left(I_{-}^{1-\alpha+[\operatorname{Re}(\alpha)]} f\right)(x) = \left(-\frac{d}{dx}\right)^{[\operatorname{Re}(\alpha)]+1}\frac{1}{\Gamma 1-\alpha+\operatorname{Re}(\alpha)])}$$
$$\int_x^\infty \frac{f(t)}{(t-x)^{\alpha-[\operatorname{Re}(\alpha)]}}\,dt, \quad (x>0) \tag{2.4}$$

respectively, where [Re(α)] is the integral part of Re (α).

In particular, for real α > 0, the operators D_{0+}^{α} and D_{-}^{α} take more simple forms.

$$(D_{0+}^{\alpha} f)(x) = \left(\frac{d}{dx}\right)^{[\alpha]+1} (I_{0+}^{1-\{\alpha\}} f)(x)$$

$$= \left(\frac{d}{dx}\right)^{[\alpha]+1} \frac{1}{\Gamma(1-\{\alpha\})} \int_0^x \frac{f(t)}{(x-t)^{\{\alpha\}}} dt, \quad (x > 0) \qquad (2.5)$$

$$(D_{-}^{\alpha} f)(x) = \left(-\frac{d}{dx}\right)^{[\alpha]+1} (I_{-}^{1-\{\alpha\}} f)(x)$$

$$= \left(-\frac{d}{dx}\right)^{[\alpha]+1} \frac{1}{\Gamma(1-\{\alpha\})} \int_x^{\infty} \frac{f(t)}{(t-x)^{\{\alpha\}}} dt, \quad (x > 0)$$

(2.6)

where [α] and {α} are integral and fractional parts of α.

For α, β, η ∈ C and x > 0 the generalized fractional calculus operators [13] are defined by

$$(I_{0+}^{\alpha,\beta,\eta} f)(x) = \frac{x^{-\alpha-\beta}}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} \; {}_2F_1(\alpha+\beta, -\eta; \alpha; 1-\frac{t}{x}) f(t)\, dt, \quad [\mathrm{Re}(\alpha) > 0]$$

(2.7)

$$(I_{0+}^{\alpha,\beta,\eta} f)(x) = \left(\frac{d}{dx}\right)^{n} (I_{0+}^{\alpha+n,\beta-n,\eta-n} f)(x),$$

$$(\mathrm{Re}(\alpha) < 0; \eta = [\mathrm{Re}(-\alpha)] + 1); \qquad (2.8)$$

$$(I_{-}^{\alpha,\beta,\eta} f)(x) = \frac{1}{\Gamma(\alpha)} \int_x^{\infty} (t-x)^{\alpha-1} t^{-\alpha-\beta}$$

$${}_2F_1(\alpha+\beta, -\eta; \alpha; 1-\frac{x}{t}) f(t)\, dt \quad (\mathrm{Re}(\alpha) > 0); \qquad (2.9)$$

$$(I_{-}^{\alpha,\beta,\eta} f)(x) = \left(-\frac{d}{dx}\right)^{n} (I_{-}^{\alpha+n,\beta-n,\eta} f)(x),$$

$$(\mathrm{Re}(\alpha) < 0; \eta = [\mathrm{Re}(-\alpha)] + 1); \qquad (2.10)$$

and

$$(D_{0+}^{\alpha,\beta,\eta} f)(x) \equiv (I_{0+}^{-\alpha,-\beta,\alpha+\eta}) f(x) = \left(\frac{d}{dx}\right)^{n} (I_{0+}^{-\alpha+n,-\beta-n,\alpha+\eta-n} f)(x)$$

$$(\mathrm{Re}(\alpha) > 0; n = [\mathrm{Re}(\alpha)] + 1); \qquad (2.11)$$

$$(D_{-}^{\alpha,\beta,\eta} f)(x) \equiv (I_{-}^{-\alpha,-\beta,\alpha+\eta}) f(x),$$

$$= \left(-\frac{d}{dx}\right)^{n} (I_{0+}^{-\alpha+n,-\beta-n,\alpha+\eta} f)(x) \quad (\mathrm{Re}(\alpha) > 0; n = [\mathrm{Re}(\alpha)] + 1). \qquad (2.12)$$

Here ${}_2F_1(a, b, c; z)$ $(a, b, c. z \in C)$ is the Gauss hypergeometric function of the series form

$$ {}_2F_1(a, b; c; z) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k} \frac{z^k}{k!}, \tag{2.13} $$

with $(a)_0 = 1, (a)_k = a(a+1)\dots(a+k-1) = \frac{\Gamma(a+k)}{\Gamma(a)} \; (k \in N)$ (2.14)

where Γ(z) is the gamma function [3, Chap.I] and **N** denotes the set of positive integers.

The series in (2.13) is convergent for | z | < 1 and | z | = 1 with Re (c-a-b) > 0 and can be analytically continued into {z ∈ C : |arg (1- z) | < π} [3, Chapter II].

Since

$$ {}_2F_1(0, b; c; z) = 1, \tag{2.15} $$

the generalized fractional calculus operators (2.7), (2.9), (2.11) and (2.12) coincide, if $\beta = -\alpha$ with the Riemann-Liouville operators (2.1) – (2.4) for Re (α) > 0:

$$ (I_{0+}^{\alpha,-\alpha,\eta} f)(x) = (I_{0+}^{\alpha} f)(x), $$

$$ (I_{-}^{\alpha,-\alpha,\eta} f)(x) = (I_{-}^{\alpha} f)(x) \tag{2.16} $$

$$ (D_{0+}^{\alpha,-\alpha,\eta} f)(x) = (D_{0+}^{\alpha} f)(x), $$

$$ (D_{-}^{\alpha,-\alpha,\eta} f)(x) = (D_{-}^{\alpha} f)(x). \tag{2.17} $$

According to the relation [3, 2.8 (4)]

$$ {}_2F_1(a, b; a; z) = (1-z)^{-b}, \tag{2.18} $$

the operators (2.7) and (2.9) coincide with the Erdelyi-Kober fractional integrals [15, § 18.1] when β = 0:

$$ (I_{0+}^{\alpha,0,\eta} f)(x) = \frac{x^{-\alpha-\eta}}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} t^{\eta} f(t)\, dt = (I_{\eta,\alpha}^{+} f)(x), \quad (\alpha, \eta \in \mathbf{C}\ \mathrm{Re}(\alpha) > 0), \tag{2.19} $$

$$ (I_{-}^{\alpha,0,\eta} f)(x) = \frac{x^{\eta}}{\Gamma(\alpha)} \int_x^{\infty} (t-x)^{\alpha-1} t^{-\eta-\alpha} f(t)\, dt \equiv (K_{\eta,\alpha}^{-} f)(x) \quad (\alpha, \eta \in \mathbf{C}, \mathrm{Re}(\alpha)>0). \tag{2.20} $$

Therefore the operators (2.7), (2.9) and (2.11), (2.12) are called 'generalized' fractional integrals and derivatives, respectively. Moreover, the operators (2.11) and (2.12) are inverse to (2.7) and (2.9):

$$ D_{0+}^{\alpha,\beta.\eta} = (I_{0+}^{\alpha,\beta,\eta})^{-1}, \; D_{-}^{\alpha,\beta,\eta} = (I_{-}^{\alpha,\beta,\eta})^{-1} \tag{2.21} $$

We also need following asymptotic behaviour of ${}_2F_1(a, b; c; z)$ at the point z = 1.

Lemma 1 For a, b, c ∈ **C** with Re (c) > 0 and z ∈ C, there hold the following asymptotic relations near z = 1:

$$ {}_2F_1(a, b; c; z) = 0(1) \quad (z \to 1-) \tag{2.22} $$

for Re (c – a – b) > 0 ;

$$ {}_2F_1(a, b; c; z) = 0((1-z)^{c-a-b}) \quad (z \to 1-) \text{ for Re } (c-a-b) < 0; \tag{2.23} $$

and $$ {}_2F_1(a, b; c; z) = 0(\log(1-z)) \quad (z \to 1-); \text{ for } c-a-b = 0,\ a, b \neq 0, -1, -2, \dots \text{ and } |\arg(z)| < \pi. \tag{2.24} $$

II LEFT-SIDED GENERALIZED FRACTIONAL INTEGRATION OF THE $\overline{\mathrm{H}}$-FUNCTION

In this and next sections we consider the $\overline{\mathrm{H}}$-function (1.1) and (1.7) with $\alpha = \alpha_{i\gamma\infty}$ and under the assumptions $\Omega > 0$ or $\Omega = 0$ where $\Omega > 0$ is given by (1.4) and it is assumed that the existence conditions of $\overline{\mathrm{H}}$-function are satisfied.

Here we consider the left sided generalized fractional integration $I_{0+}^{\alpha,\beta,\eta}$ defined by (2.7).

Theorem 1. Let $\alpha, \beta, \eta \in \mathbf{C}$ with Re $(\alpha) > 0$, Re $(\beta) \neq$ Re (η). Let the constants $a_j, b_j \in \mathbf{C}$, $\alpha_j, \beta_j > 0$ ($j = 1,\ldots, P$; $j = 1,\ldots,Q$) and $\lambda \in \mathbf{C}$, $\sigma_1, \sigma_2 > 0$ and

$$\sigma_1 \min_{1\le j\le M_1} \mathrm{Re}\left(\frac{f_j}{F_j}\right) + \sigma_2 \min_{1\le j\le M} \mathrm{Re}\left(\frac{b_j}{\beta_j}\right) + \mathrm{Re}\,(\lambda) + \min\,[0, \mathrm{Re}\,(\eta-\beta)] > 0 \qquad (3.1)$$

Then the generalized fractional integral $I_{0+}^{\alpha,\beta,\eta}$ of the product of $\overline{\mathrm{H}}$-functions with $S_V^U[\delta\, t^\rho]$ exists and the following relation holds:

$$\left(I_{0+}^{\alpha,\beta,\eta} t^{\lambda-1}\, \overline{H}_{P_1,Q_1}^{M_1,N_1}\left[w_1 t^{\sigma_1}\left|\begin{matrix}(e_j,E_j;\varepsilon_j)_{1,N_1},(e_j,E_j)_{N_1+1,P_1}\\(f_j,F_j)_{1,M_1},(f_j,F_j;\Im_j)_{M_1+1,Q_1}\end{matrix}\right.\right]\right.$$

$$\left.\overline{H}_{P,Q}^{M,N}\left[w_2 t^{\sigma_2}\left|\begin{matrix}(a_j,\alpha_j;A_j)_{1,N},(a_j,\alpha_j)_{N+1,P}\\(b_j,\beta_j)_{1,M},(b_j,\beta_j;B_j)_{M+1,Q}\end{matrix}\right.\right] S_V^U[\delta\, t^\rho]\right)(x)$$

$$= x^{\lambda-\beta-1}\sum_{R=0}^{[V/U]}\frac{(-V)_{UR}}{R!}A(V,R)\,\delta^R x^{\rho R}\sum_{\upsilon=1}^{M_1}\sum_{p=0}^{\infty}\overline{\theta}\left(S_{P,\upsilon}\right)$$

$$\left(w_1 x^{\sigma_1}\right)^{S_{P,\upsilon}}\overline{H}_{P+2,Q+2}^{M,N+2}\left[w_2 x^{\sigma_2}\left|\begin{matrix}\left(1-\lambda-\rho R-\sigma_1 S_{p,\upsilon},\sigma_2;1\right)\\(b_j,\beta_j)_{1,M},(b_j,\beta_j;B_j)_{M+1,Q}\end{matrix}\right.\right.$$

$$\left.\begin{matrix}\left(1-\lambda-\eta+\beta-\rho R-\sigma_1 S_{p,\upsilon},\sigma_2;1\right)(a_j,\alpha_j;A_j)_{1,N},(a_j,\alpha_j)_{N+1,P}\\\left(1-\lambda+\beta-\rho R-\sigma_1 S_{p,\upsilon},\sigma_2;1\right)\left(1-\lambda-\alpha-\eta-\rho R-\sigma_1 S_{p,\upsilon},\sigma_2;1\right)\end{matrix}\right]$$

(3.2)

Proof. By (2.7), we have

$$\left(I_{0+}^{\alpha,\beta,\eta} t^{\lambda-1}\,\overline{H}_{P_1,Q_1}^{M_1,N_1}\left[w_1 t^{\sigma_1}\right]\overline{H}_{P,Q}^{M,N}\left[w_2 t^{\sigma_2}\right]S_V^U[\delta\, t^\rho]\right)(x)$$

$$= \frac{x^{-\alpha-\beta}}{\Gamma(\alpha)}\int_0^x (x-t)^{\alpha-1}\, t^{\varpi}\,{}_2F_1(\alpha+\beta,-\eta;\alpha;1-\frac{t}{x})$$

$$\overline{H}_{P_1,Q_1}^{M_1,N_1}\left[w_1 t^{\sigma_1}\right]\overline{H}_{P,Q}^{M,N}\left[w_2 t^{\sigma_2}\right]S_V^U[\delta\, t^\rho]\,dt \qquad (3.3)$$

According to (2.22), (2.23) the integrand in (3.3) for any $x > 0$ has the asymptotic estimate at zero

$$(x-t)^{\alpha-1} t^{\omega}\, {}_2F_1\left(\alpha+\beta,-\eta;\alpha;\frac{t}{x}\right) \bar{H}^{M_1,N_1}_{P_1,Q_1}\left[w_1 t^{\sigma_1}\right]\bar{H}^{M,N}_{P,Q}\left[w_2 t^{\sigma_2}\right] S_V^U[\delta t^{\rho}]$$

$$= O\left(t^{\lambda+\sigma_1 e^*+\sigma_2 f^*+\min[0,\,\mathrm{Re}(\eta+\beta)]-1}\right)(t\to+0)$$

$$= O\left(t^{\lambda+\sigma_1 e^*+\sigma_2 f^*+\min[0,\,\mathrm{Re}(\eta+\beta)]-1}[\log(t)]^{N^*}\right),(t\to+0)$$

Here $e^* = \min_{1\le j\le M_1}\left[\frac{\mathrm{Re}(f_j)}{F_j}\right]$, $f^* = \min_{1\le j\le M}\left[\frac{\mathrm{Re}(b_j)}{\beta_j}\right]$ and N^* is the order of one of the poles $b_{j\ell} = \frac{-b_j-\ell}{\beta_j}$ (j = 1,…,M; ℓ = 0,1,2,…) to which some other poles of $\Gamma(b_j+\beta_j s)$ (j =1,…,M) coincide. Therefore the condition (3.1) ensures the existence of the integral (3.3).

Applying (1.1), (1.7), (1.11), making the change of variable t = x τ, changing the order of summation and integration and taking into account the formula [11, § 2.21.1.11]

$$\int_0^x t^{\alpha-1}(x-t)^{c-1}\,{}_2F_1(a,b;c;1-\frac{t}{x})dt = \frac{\Gamma(c)\Gamma(\alpha)\Gamma(\alpha+c-a-b)}{\Gamma(\alpha+c-a)\Gamma(\alpha+c-b)}x^{\alpha+c-1},$$ (a, b, c, α ∈ **C**, Re (α) > 0, Re (c) > 0,

Re (α+c – a – b > 0). (3.4)

We obtain

$$\left(I_{0+}^{\alpha,\beta,\eta} t^{\lambda-1}\bar{H}^{M_1,N_1}_{P_1,Q_1}\left[w_1 t^{\sigma_1}\right]\bar{H}^{M,N}_{P,Q}\left[w_2 t^{\sigma_2}\right]S_V^U[\delta t^{\rho}]\right)(x)$$

$$= \frac{x^{-\alpha-\beta}}{\Gamma(\alpha)}\frac{1}{2\pi i}\int_L \bar{\phi}(\xi)(w_2)^{\xi}d\xi \sum_{R=0}^{[V/U]}\frac{(-V)_{UR}}{R!}A_{(V,R)}\delta^R \sum_{\upsilon=1}^{M_1}\sum_{p=0}^{\infty}\bar{\theta}(S_{P,\upsilon})(w_1)^{S_{P,\upsilon}}\int_0^x t^{\lambda+\rho R+\sigma_1 S_{p,\upsilon}+\sigma_2\xi-1}$$

$$(x-t)^{\alpha-1}\,{}_2F_1(\alpha+\beta,-\eta;\alpha;1-\frac{t}{x})\,dt,$$

$$= x^{\lambda-\beta-1}\sum_{R=0}^{[V/U]}\frac{(-V)_{UR}}{R!}A_{V,R}\left(\delta t^{\rho}\right)^R \sum_{\upsilon=1}^{M_1}\sum_{p=0}^{\infty}\bar{\theta}(S_{P,\upsilon})\left(w_1 t^{\sigma_1}\right)^{S_{P,\upsilon}}$$

$$.\frac{1}{2\pi i}\int_{\alpha}\bar{\phi}(\xi)\frac{\Gamma(\lambda+\rho R+\sigma_1 S_{p,\upsilon}+\sigma_2\xi)}{\Gamma(\lambda-\beta+\rho R+\sigma_1 S_{p,\upsilon}+\sigma_2\xi)} \frac{\Gamma(\lambda+\eta-\beta+\rho R+\sigma_1 S_{p,\upsilon}+\sigma_2\xi)}{\Gamma(\lambda+\alpha+\eta+\rho R+\sigma_1 S_{p,\upsilon}+\sigma_2\xi\eta)}x^{\sigma_2\xi}d\xi$$ and in accordance with (1.1), we obtain (3.2) which completes the proof of Theorem 1.

Corollary 1.1 Let α ∈ **C** with Re (α) > 0 and let the constants λ ∈ **C**, $\sigma_1,\sigma_2>0$ and

$$\sigma_1 \min_{1\le j\le M_1}\mathrm{Re}\left(\frac{f_j}{F_j}\right)+\sigma_2\min_{1\le j\le M}\mathrm{Re}\left(\frac{b_j}{\beta_j}\right)+\mathrm{Re}(\lambda)>0 \qquad (3.5)$$

then the Riemann-Liouville fractional integral I_{0+}^{α} of the product of $\bar{H}$-functions with $S_V^U[\delta t^{\rho}]$ exists and the following relation holds:

$$\left(I_{0+}^{\alpha} t^{\lambda-1}\bar{H}^{M_1,N_1}_{P_1,Q_1}\left[w_1 t^{\sigma_1}\right]\bar{H}^{M,N}_{P,Q}\left[w_2 t^{\sigma_2}\right]S_V^U[\delta t^{\rho}]\right)(x)$$

$$= x^{\lambda+\alpha-1}\sum_{R=0}^{[V/U]}\frac{(-V)_{UR}}{R!}A_{(V,R)}\left(\delta x^{\rho}\right)^R\sum_{\upsilon=1}^{M_1}\sum_{p=0}^{\infty}\bar{\theta}(S_{P,\upsilon})$$

$$\left(w_1 x^{\sigma_1}\right)^{S_{P,\upsilon}} \overline{H}^{M,N+1}_{P+1,Q+1}\left[w_2 x^{\sigma_2}\left|\begin{matrix}(1-\lambda-\rho R-\sigma_1 S_{p,\upsilon},\sigma_2;1) & (a_j,\alpha_j;A_j)_{1,N},(a_j,\alpha_j)_{N+1,P}\\ (b_j,\beta_j)_{1,M},(b_j,\beta_j;B_j)_{M+1,Q} & (1-\lambda-\alpha-\rho R-\sigma_1 S_{p,\upsilon},\sigma_2;1)\end{matrix}\right.\right] \tag{3.6}$$

Corollary 1.2 Let α, η, ∈ **C** with Re (α) > 0 and let the constants λ ∈ **C**, $\sigma_1, \sigma_2 > 0$ satisfy and

$$\sigma_1 \min_{1\le j\le M_1} \mathrm{Re}\left(\frac{f_j}{F_j}\right)+\sigma_2 \min_{1\le j\le M} \mathrm{Re}\left(\frac{b_j}{\beta_j}\right) + \mathrm{Re}\,(\lambda)+\min\,[0,\, \mathrm{Re}\,(\eta)] > 0$$

(3.7)

then the Erdélyi-kober fractional integral $I^{+}_{\eta,\alpha}$ of the product of $\overline{H}$ -functions with $S_V^U[\delta\, t^{\rho}]$ exists and the following relation holds:

$$\left(I^{+}_{\eta,\alpha} t^{\lambda-1}\, \overline{H}^{M_1,N_1}_{P_1,Q_1}\left[w_1 t^{\sigma_1}\right]\overline{H}^{M,N}_{P,Q}\left[w_2 t^{\sigma_2}\right]S_V^U[\delta\, t^{\rho}]\right)(x)$$

$$= x^{\lambda-1}\sum_{R=0}^{[V/U]}\frac{(-V)_{UR}}{R!}A_{(V,R)}\left(\delta x^{\rho}\right)^R \sum_{\upsilon=1}^{M_1}\sum_{p=0}^{\infty}\overline{\theta}\left(S_{P,\upsilon}\right)$$

$$\left(w_1 x^{\sigma_1}\right)^{S_{P,\upsilon}} \overline{H}^{M,N+1}_{P+1,Q+1}\left[w_2 x^{\sigma_2}\left|\begin{matrix}(1-\lambda-\eta-\rho R-\sigma_1 S_{p,\upsilon},\sigma_2;1) & (a_j,\alpha_j;A_j)_{1,N},(a_j,\alpha_j)_{N+1,P}\\ (b_j,\beta_j)_{1,M},(b_j,\beta_j;B_j)_{M+1,Q} & (1-\lambda-\alpha-\eta-\rho R-\sigma_1 S_{p,\upsilon},\sigma_2;1)\end{matrix}\right.\right] \tag{3.8}$$

IV RIGHT-SIDED GENERALIZED FRACTIONAL INTEGRATION OF THE $\overline{H}$ -FUNCTION

In this section we consider the right sided generalized fractional integration $I_{-}^{\alpha,\beta,\eta}$ defined by (2.9).

Theorem 2 Let α, β, η ∈ **C** with Re (α) > 0, Re (β) ≠ Re (η). Let the constants λ ∈ **C**, $\sigma_1, \sigma_2 > 0$ and

$$\sigma_1 \max_{1\le j\le N_1}\left[\frac{\mathrm{Re}(e_j)-1}{E_j}\right]+\sigma_2 \max_{1\le j\le N}\left[\frac{\mathrm{Re}(a_j)-1}{\alpha_j}\right] + \mathrm{Re}\,(\lambda) < \min\,[\mathrm{Re}\,(\beta),\, \mathrm{Re}\,(\eta)] \tag{4.1}$$

then the generalized fractional integral $I_{-}^{\alpha,\beta,\eta}$ of the product of $\overline{H}$ -functions with $S_V^U[\delta\, t^{\rho}]$ exists and the following relation holds:

$$\left(I_{-}^{\alpha,\beta,\eta} t^{\lambda-1}\, \overline{H}^{M_1,N_1}_{P_1,Q_1}\left[w_1 t^{\sigma_1}\right]\overline{H}^{M,N}_{P,Q}\left[w_2 t^{\sigma_2}\right]S_V^U[\delta\, t^{\rho}]\right)(x)$$

$$= x^{\lambda-\beta-1}\sum_{R=0}^{[V/U]}\frac{(-V)_{UR}}{R!}A_{(V,R)}\left(\delta x^{\rho}\right)^R \sum_{\upsilon=1}^{M_1}\sum_{p=0}^{\infty}\overline{\theta}\left(S_{P,\upsilon}\right)\left(w_1 x^{\sigma_1}\right)^{S_{P,\upsilon}}$$

$$\overline{H}^{M+2,N}_{P+2,Q+2}\left[w_2 x^{\sigma_2}\left|\begin{matrix}(a_j,\alpha_j;A_j)_{1,N},(a_j,\alpha_j)_{N+1,P}\\ (1-\lambda+\beta-\rho R-\sigma_1 S_{p,\upsilon},\sigma_2)\end{matrix}\right.\right.$$

$$\left.\begin{matrix}(1-\lambda-\rho R-\sigma_1 S_{p,\upsilon},\sigma_2)(1-\lambda+\alpha+\beta+\eta-\rho R-\sigma_1 S_{p,\upsilon},\sigma_2)\\ (1-\lambda+\eta-\rho R-\sigma_1 S_{p,\upsilon},\sigma_2)(b_j,\beta_j)_{1,M},(b_j,\beta_j;B_j)_{M+1,Q}\end{matrix}\right] \tag{4.2}$$

Proof: By (2.9), we have

$$\left(I_{-}^{\alpha,\beta,\eta} t^{\lambda-1}\bar{H}_{P_1,Q_1}^{M_1,N_1}\left[w_1 t^{\sigma_1}\right]\bar{H}_{P,Q}^{M,N}\left[w_2 t^{\sigma_2}\right]S_V^U[\delta t^{\rho}]\right)(x)$$

$$= \frac{1}{\Gamma(\alpha)}\int_x^{\infty} (t-x)^{\alpha-1} t^{\lambda-\alpha-\beta-1} {}_2F_1(\alpha+\beta,-\eta;\alpha;1-\frac{x}{t}) \bar{H}_{P_1,Q_1}^{M_1,N_1}\left[w_1 t^{\sigma_1}\right]\bar{H}_{P,Q}^{M,N}\left[w_2 t^{\sigma_2}\right]S_V^U[\delta t^{\rho}]\, dt \tag{4.3}$$

Due to (2.22) and (2.23) the integrand in (4.3) for any x > 0 has the asymptotic estimate at infinity

$$(t-x)^{\alpha-1} t^{\lambda-\alpha-\beta-1} {}_2F_1\left(\alpha+\beta,-\eta;\alpha;1-\frac{x}{t}\right)$$

$$\bar{H}_{P_1,Q_1}^{M_1,N_1}\left[w_1 t^{\sigma_1}\right]\bar{H}_{P,Q}^{M,N}\left[w_2 t^{\sigma_2}\right]S_V^U[\delta t^{\rho}] = O\left(t^{\lambda+\sigma_1 e_1^*+\sigma_2 f_1^*-\min[\mathrm{Re}(\beta),\mathrm{Re}(\eta)]-1}\right), (t\to+\infty)$$

$$= O\left(t^{\lambda+\sigma_1 e_1^*+\sigma_2 f_1^*-\min[\mathrm{Re}(\beta),\mathrm{Re}(\eta)]-1}[\log(t)]^{N^*}\right), (t\to+\infty)$$

Here $e_1^* = \max_{1\le j\le N_1}\left[\frac{\mathrm{Re}(e_j)-1}{E_j}\right]$, $f_1^* = \max_{1\le j\le N}\left[\frac{\mathrm{Re}(a_j)-1}{\alpha_j}\right]$ and N^* is the order of one of the poles $a_i k = \frac{1-a_i+k}{\alpha_i}$ (i = 1,…,N ; k = 0,1,2,…) to which some other poles of $\Gamma(1- a_i- \alpha_i s)$ (i = 1,2,…,N) coincide.

Therefore the condition (4.1) ensures the existence of the integral (4.3).

Applying (1.2) making the change $t = \frac{1}{\tau}$ and using (3.4), we obtain

$$\left(I_{-}^{\alpha,\beta,\eta} t^{\lambda-1}\bar{H}_{P_1,Q_1}^{M_1,N_1}\left[w_1 t^{\sigma_1}\right]\bar{H}_{P,Q}^{M,N}\left[w_2 t^{\sigma_2}\right]S_V^U[\delta t^{\rho}]\right)(x)$$

$$= \frac{1}{\Gamma(\alpha)}\int_{1/x}^{\infty}\left(t-\frac{1}{x}\right)^{\alpha-1} t^{\lambda-\alpha-\beta-1}$$

$${}_2F_1(\alpha+\beta,-\eta;\alpha;1-\frac{1}{tx})\sum_{R=0}^{[V/U]}\frac{(-V)_{UR}}{R!}A_{(V,R)}\delta^R \sum_{\upsilon=1}^{M_1}\sum_{p=0}^{\infty}\bar{\theta}\left(S_{P,\upsilon}\right)\left(w_1\right)^{S_{P,\upsilon}}\frac{1}{2\pi i}\int_L\bar{\phi}(\xi)\left(w_2 t^{\sigma_2}\right)^{\xi}\, d\xi\, dt$$

$$= \frac{x^{1-\alpha}}{\Gamma(\alpha)}\int_0^x (x-\tau)^{\alpha-1}\tau^{\alpha+\beta-\lambda-\rho R-\sigma_1 S_{p,\upsilon}-\sigma_2\xi-1}$$

$${}_2F_1(\alpha+\beta,-\eta;\alpha;1-\frac{\tau}{x})\,d\tau\sum_{R=0}^{[V/U]}\frac{(-V)_{UR}}{R!}A_{(V,R)}\delta^R \sum_{\upsilon=1}^{M_1}\sum_{p=0}^{\infty}\bar{\theta}\left(S_{P,\upsilon}\right)\left(w_1\right)^{S_{P,\upsilon}}\frac{1}{2\pi i}\int_L\bar{\phi}(\xi)\left(w_2\right)^{\xi}\, d\xi$$

$$= x^{\lambda-\beta-1}\sum_{R=0}^{[V/U]}\frac{(-V)_{UR}}{R!}A_{(V,R)}\left(\delta t^{\rho}\right)^R\sum_{\upsilon=1}^{M_1}\sum_{p=0}^{\infty}\bar{\theta}\left(S_{P,\upsilon}\right)$$

$$\left(w_1 t^{\sigma_1}\right)^{S_{P,\upsilon}}\frac{1}{2\pi i}\int_L\bar{\phi}(\xi)\left(w_2\right)^{\xi}\frac{\Gamma(1-\lambda+\beta-\rho R-\sigma_1 S_{p,\upsilon}-\sigma_2\xi)}{\Gamma(1-\lambda-\rho R-\sigma_1 S_{p,\upsilon}-\sigma_2\xi)}$$

$$\frac{\Gamma(1-\lambda+\eta-\rho R-\sigma_1 S_{p,\upsilon}-\sigma_2\xi)}{\Gamma(1-\lambda+\alpha+\beta+\eta-\rho R-\sigma_1 S_{p,\upsilon}-\sigma_2\xi)}\, x^{\sigma_2\xi}\, d\xi$$

and in accordance with (1.1), we obtain (4.2) which completes the proof of Theorem 2.

Corollary 2.1 Let α ∈ **C**, with Re (α) > 0 and let the constants λ ∈ **C**, $\sigma_1, \sigma_2 > 0$ satisfy and

$$\sigma_1 \max_{1 \le j \le N_1} \left[\frac{\mathrm{Re}(e_j)-1}{E_j} \right] + \sigma_2 \max_{1 \le j \le N} \left[\frac{\mathrm{Re}(a_j)-1}{\alpha_j} \right] + \mathrm{Re}(\lambda) + \mathrm{Re}(\alpha) < 0 \qquad (4.5)$$

Then the Riemann-Liouville fractional integral I_{-}^{α} of the product of $\overline{H}$ -functions with $S_V^U[\delta t^{\rho}]$ exist and the following relation hold:

$$\left(I_{-}^{\alpha} t^{\lambda-1} \overline{H}_{P_1,Q_1}^{M_1,N_1} \left[w_1 t^{\sigma_1} \right] \overline{H}_{P,Q}^{M,N} \left[w_2 t^{\sigma_2} \right] S_V^U[\delta t^{\rho}] \right)(x)$$

$$= x^{\lambda+\alpha-1} \sum_{R=0}^{[V/U]} \frac{(-V)_{UR}}{R!} A_{(V,R)} \left(\delta x^{\rho}\right)^R \sum_{\upsilon=1}^{M_1} \sum_{p=0}^{\infty} \bar{\theta}\left(S_{P,\upsilon}\right)$$

$$\left(w_1 x^{\sigma_1}\right)^{S_{P,\upsilon}} \overline{H}_{P+1,Q+1}^{M+1,N} \left[w_2 x^{\sigma_2} \middle| \begin{matrix} (a_j, \alpha_j; A_j)_{1,N}, (a_j, \alpha_j)_{N+1,P} \\ \left(1-\lambda-\alpha-\rho R-\sigma_1 S_{p,\upsilon}, \sigma_2\right) \end{matrix} \right.$$

$$\left. \begin{matrix} \left(1-\lambda-\rho R-\sigma_1 S_{p,\upsilon}, \sigma_2\right) \\ (b_j, \beta_j)_{1,M}, (b_j, \beta_j; B_j)_{M+1,Q} \end{matrix} \right]. \qquad (4.6)$$

Corollary 2.2 Let α ∈ **C**, with Re (α) > 0 and let the constants λ ∈ **C**, $\sigma_1, \sigma_2 > 0$ satisfy and

$$\sigma_1 \max_{1 \le j \le N_1} \left[\frac{\mathrm{Re}(e_j)-1}{E_j} \right] + \sigma_2 \max_{1 \le j \le N} \left[\frac{\mathrm{Re}(a_j)-1}{\alpha_j} \right] + \mathrm{Re}(\lambda) + \mathrm{Re}(\eta) < 0 \qquad (4.7)$$

Then the Erdelyi-Kober fractional integral $K_{\eta,\alpha}^{-}$ of the product of $\overline{H}$ -functions with $S_V^U[\delta t^{\rho}]$ exist and the following relation hold:

$$\left(K_{\eta,\alpha}^{-} t^{\lambda-1} \overline{H}_{P_1,Q_1}^{M_1,N_1} \left[w_1 t^{\sigma_1} \right] \overline{H}_{P,Q}^{M,N} \left[w_2 t^{\sigma_2} \right] S_V^U[\delta t^{\rho}] \right)(x)$$

$$= x^{\lambda-1} \sum_{R=0}^{[V/U]} \frac{(-V)_{UR}}{R!} A_{(V,R)} \left(\delta x^{\rho}\right)^R \sum_{\upsilon=1}^{M_1} \sum_{p=0}^{\infty} \bar{\theta}\left(S_{P,\upsilon}\right) \left(w_1 x^{\sigma_1}\right)^{S_{P,\upsilon}}$$

$$\overline{H}_{P+1,Q+1}^{M+1,N} \left[w_2 x^{\sigma_2} \middle| \begin{matrix} (a_j, \alpha_j; A_j)_{1,N}, (a_j, \alpha_j)_{N+1,P} \\ \left(1-\lambda+\eta-\alpha-\rho R-\sigma_1 S_{p,\upsilon}, \sigma_2\right) \end{matrix} \begin{matrix} \left(1-\lambda+\alpha+\beta+\eta-\rho R-\sigma_1 S_{p,\upsilon}, \sigma_2\right) \\ (b_j, \beta_j)_{1,M}, (b_j, \beta_j; B_j)_{M+1,Q} \end{matrix} \right] \qquad (4.8)$$

V LEFT-SIDED GENERALIZED FRACTIONAL DIFFERENTIATION OF THE $\overline{H}$ -FUNCTION

Now we treat the left-sided generalized fractional derivative $D_{0+}^{\alpha,\beta,\eta}$ given by (2.11).

Theorem 3. Let α, β, η ∈ **C** with Re (α) > 0, Re (α+β+η) ≠ 0. Let the constants a_j, b_j ∈ **C**, α_j, β_j > 0 (j = 1,…, P; j = 1,…,Q) and λ ∈ **C**, $\sigma_1, \sigma_2 > 0$ and

$$\sigma_1 \min_{1\le j\le M_1} \mathrm{Re}\left(\frac{f_j}{F_j}\right)+\sigma_2 \min_{1\le j\le M} \mathrm{Re}\left(\frac{b_j}{\beta_j}\right) + \mathrm{Re}\,(\lambda) + \min\,[0, \mathrm{Re}\,(\alpha+\beta+\eta)] > 0 \quad (5.1)$$

Then the generalized fractional derivative $D_{0+}^{\alpha,\beta,\eta}$ of the product of $\overline{\mathrm{H}}$ -functions with $S_V^U[\delta t^{\rho}]$ exists and the following relation holds:

$$\left(D_{0+}^{\alpha,\beta,\eta} t^{\lambda-1}\, \overline{\mathrm{H}}_{P_1,Q_1}^{M_1,N_1}\left[w_1 t^{\sigma_1}\right] \overline{\mathrm{H}}_{P,Q}^{M,N}\left[w_2 t^{\sigma_2}\right] S_V^U[\delta t^{\rho}]\right)(x)$$

$$= x^{\lambda+\beta-1} \sum_{R=0}^{[V/U]} \frac{(-V)_{UR}}{R!} A_{(V,R)} \left(\delta x^{\rho}\right)^R \sum_{\upsilon=1}^{M_1}\sum_{p=0}^{\infty} \bar{\theta}\left(S_{p,\upsilon}\right)$$

$$\left(w_1 x^{\sigma_1}\right)^{S_{p,\upsilon}} \overline{H}_{P+2,Q+2}^{M,N+2}\left[w_2 x^{\sigma_2} \middle| \begin{matrix} \left(1-\lambda-\rho R-\sigma_1 S_{p,\upsilon},\sigma_2;1\right) \\ (b_j,\beta_j)_{1,M},(b_j,\beta_j;B_j)_{M+1,Q} \end{matrix}\right.$$

$$\left.\begin{matrix} \left(1-\lambda-\alpha-\beta-\eta-\rho R-\sigma_1 S_{p,\upsilon},\sigma_2;1\right)(a_j,\alpha_j;A_j)_{1,N},(a_j,\alpha_j)_{N+1,P} \\ \left(1-\lambda-\beta-\rho R-\sigma_1 S_{p,\upsilon},\sigma_2;1\right)\left(1-\lambda-\eta-\rho R-\sigma_1 S_{p,\upsilon},\sigma_2;1\right) \end{matrix}\right] \quad (5.2)$$

Proof. Let n = [Re (α)] +1. From (2.11) we have

$$\left(D_{0+}^{\alpha,\beta,\eta} t^{\lambda-1}\, \overline{\mathrm{H}}_{P_1,Q_1}^{M_1,N_1}\left[w_1 t^{\sigma_1}\right] \overline{\mathrm{H}}_{P,Q}^{M,N}\left[w_2 t^{\sigma_2}\right] S_V^U[\delta t^{\rho}]\right)(x)$$

$$= \left(\frac{d}{dx}\right)^n \left(I_{0+}^{-\alpha+n,-\beta-n,\alpha+\eta-n} t^{\lambda-1}\, \overline{\mathrm{H}}_{P_1,Q_1}^{M_1,N_1}\left[w_1 t^{\sigma_1}\right] \overline{\mathrm{H}}_{P,Q}^{M,N}\left[w_2 t^{\sigma_2}\right] S_V^U[\delta t^{\rho}]\right) \quad (5.3)$$

which exists according to Theorem 1 with α, β and η being replaced by - α + n, - β - n and α + η - n respectively. Then we obtain

$$\left(D_{0+}^{\alpha,\beta,\eta} t^{\lambda-1}\, \overline{\mathrm{H}}_{P_1,Q_1}^{M_1,N_1}\left[w_1 t^{\sigma_1}\right] \overline{\mathrm{H}}_{P,Q}^{M,N}\left[w_2 t^{\sigma_2}\right] S_V^U[\delta t^{\rho}]\right)(x)$$

$$= \left(\frac{d}{dx}\right)^n x^{\lambda+\beta+n-1} \sum_{R=0}^{[V/U]} \frac{(-V)_{UR}}{R!} A_{(V,R)} \left(\delta x^{\rho}\right)^R \sum_{\upsilon=1}^{M_1}\sum_{p=0}^{\infty} \bar{\theta}\left(S_{p,\upsilon}\right)$$

$$\left(w_1 x^{\sigma_1}\right)^{S_{p,\upsilon}} \overline{H}_{P+2,Q+2}^{M,N+2}\left[w_2 x^{\sigma_2} \middle| \begin{matrix} \left(1-\lambda-\rho R-\sigma_1 S_{p,\upsilon},\sigma_2;1\right) \\ (b_j,\beta_j)_{1,M},(b_j,\beta_j;B_j)_{M+1,Q} \end{matrix}\right.$$

$$\left.\begin{matrix} \left(1-\lambda-\alpha-\beta-\eta-\rho R-\sigma_1 S_{p,\upsilon},\sigma_2;1\right)(a_j,\alpha_j;A_j)_{1,N},(a_j,\alpha_j)_{N+1,P} \\ \left(1-\lambda-\beta-n-\rho R-\sigma_1 S_{p,\upsilon},\sigma_2;1\right)\left(1-\lambda-\eta-\rho R-\sigma_1 S_{p,\upsilon},\sigma_2;1\right) \end{matrix}\right] \quad (5.4)$$

on differentiating n times and the relation $n\Gamma(n)=\Gamma(n+1)$ imply (5.2) which completes the proof of the theorem.

Corollary 3.1 Let α ∈ **C** with Re (α) > 0 and let the constants and λ ∈ **C**, $\sigma_1,\sigma_2>0$ satisfy the conditions in (3.8).

Then the Riemann-Liouville fractional derivative D_{0+}^{α} of the product of $\overline{\mathrm{H}}$ -functions with $S_V^U[\delta t^{\rho}]$ exists and the following relation holds:

$$\left(D_{0+}^{\alpha} t^{\lambda-1}\, \overline{\mathrm{H}}_{P_1,Q_1}^{M_1,N_1}\left[w_1 t^{\sigma_1}\right] \overline{\mathrm{H}}_{P,Q}^{M,N}\left[w_2 t^{\sigma_2}\right] S_V^U[\delta t^{\rho}]\right)(x)$$

$$= x^{\lambda-\alpha-1}\sum_{R=0}^{[V/U]}\frac{(-V)_{UR}}{R!}A_{(V,R)}\left(\delta x^{\rho}\right)^{R}\sum_{\upsilon=1}^{M_1}\sum_{p=0}^{\infty}\bar{\theta}\left(S_{p,\upsilon}\right)$$

$$\left(w_1x^{\sigma_1}\right)^{S_{p,\upsilon}}\overline{H}^{M,N+1}_{P+1,Q+1}\left[w_2x^{\sigma_2}\left|\begin{matrix}\left(1-\lambda-\rho R-\sigma_1 S_{p,\upsilon},\sigma_2;1\right) & (a_j,\alpha_j;A_j)_{1,N},(a_j,\alpha_j)_{N+1,P}\\ (b_j,\beta_j)_{1,M},(b_j,\beta_j;B_j)_{M+1,Q} & \left(1-\lambda+\alpha-\rho R-\sigma_1 S_{p,\upsilon},\sigma_2;1\right)\end{matrix}\right.\right] \quad (5.2)$$

VI RIGHT-SIDED GENERALIZED FRACTIONAL DIFFERENTIATION OF THE $\overline{\mathrm{H}}$ -FUNCTION

Theorem 4. Let α, β, η ∈ **C** with Re (α) > 0, Re (α + β + η) + Re(α) + 1 ≠ 0. Let the constants λ ∈ **C**, $\sigma_1,\sigma_2>0$ satisfy and

$$\sigma_1\max_{1\le j\le N_1}\left[\frac{\operatorname{Re}\left(e_j\right)-1}{E_j}\right]+\sigma_2\max_{1\le j\le N}\left[\frac{\operatorname{Re}\left(a_j\right)-1}{\alpha_j}\right]\qquad+\operatorname{Re}(\lambda)\ -1+\max\left[\operatorname{Re}(\beta),\left\{\operatorname{Re}(\alpha)+1\right\}-\operatorname{Re}(\alpha+\eta)\right]<0$$

(6.1)

Then the generalized fractional derivative $D_{-}^{\alpha,\beta,\eta}$ of the product of $\overline{\mathrm{H}}$ -functions with $S_V^U[\delta t^{\rho}]$ exists and the following relation holds:

$$\left(D_{-}^{\alpha,\beta,\eta}t^{\lambda-1}\overline{H}^{M_1,N_1}_{P_1,Q_1}\left[w_1t^{\sigma_1}\right]\overline{H}^{M,N}_{P,Q}\left[w_2t^{\sigma_2}\right]S_V^U[\delta t^{\rho}]\right)(x)$$

$$=(-1)^{\left[\operatorname{Re}(\alpha)+1\right]}x^{\lambda+\beta-1}\sum_{R=0}^{[V/U]}\frac{(-V)_{UR}}{R!}A_{(V,R)}\left(\delta x^{\rho}\right)^{R}\sum_{\upsilon=1}^{M_1}\sum_{p=0}^{\infty}\bar{\theta}\left(S_{p,\upsilon}\right)$$

$$\left(w_1x^{\sigma_1}\right)^{S_{p,\upsilon}}\overline{H}^{M+2,N}_{P+2,Q+2}\left[w_2x^{\sigma_2}\left|\begin{matrix}(a_j,\alpha_j;A_j)_{1,N},(a_j,\alpha_j)_{N+1,P}\\ \left(1-\lambda+\alpha+\eta-\rho R-\sigma_1S_{p,\upsilon},\sigma_2;1\right)\end{matrix}\right.\right.$$

$$\left.\begin{matrix}\left(1-\lambda-\rho R-\sigma_1S_{p,\upsilon},\sigma_2;1\right)\left(1-\lambda-\beta+\eta-\rho R-\sigma_1S_{p,\upsilon},\sigma_2;1\right)\\ \left(1-\lambda-\beta-\rho R-\sigma_1S_{p,\upsilon},\sigma_2;1\right)(b_j,\beta_j)_{1,M},(b_j,\beta_j;B_j)_{M+1,Q}\end{matrix}\right] \quad (6.2)$$

Proof. Let n = [Re (α)] + 1. From (2.12) we have

$$\left(D_{-}^{\alpha,\beta,\eta}t^{\lambda-1}\overline{H}^{M_1,N_1}_{P_1,Q_1}\left[w_1t^{\sigma_1}\right]\overline{H}^{M,N}_{P,Q}\left[w_2t^{\sigma_2}\right]S_V^U[\delta t^{\rho}]\right)(x)$$

$$=\left(-\frac{d}{dx}\right)^{n}\left(I_{-}^{-\alpha+n,-\beta-n,\alpha+\eta}t^{\lambda-1}\ \overline{H}^{M_1,N_1}_{P_1,Q_1}\left[w_1t^{\sigma_1}\right]\overline{H}^{M,N}_{P,Q}\left[w_2t^{\sigma_2}\right]S_V^U[\delta t^{\rho}]\right)$$

(6.3)

which exists according to Theorem 2 with α, β and η being replaced by - α + n, - β - n and α + η respectively. Then we obtain

$$\left(D_{-}^{\alpha,\beta,\eta}t^{\lambda-1}\overline{H}^{M_1,N_1}_{P_1,Q_1}\left[w_1t^{\sigma_1}\right]\overline{H}^{M,N}_{P,Q}\left[w_2t^{\sigma_2}\right]S_V^U[\delta t^{\rho}]\right)(x)$$

$$=\left(-\frac{d}{dx}\right)^{n}x^{\lambda+\beta+n-1}\sum_{R=0}^{[V/U]}\frac{(-V)_{UR}}{R!}A_{(V,R)}\left(\delta x^{\rho}\right)^{R}\sum_{\upsilon=1}^{M_1}\sum_{p=0}^{\infty}\bar{\theta}\left(S_{p,\upsilon}\right)$$

$$\left(w_1 x^{\sigma_1}\right)^{S_{p,\upsilon}} \overline{H}^{M+2,N}_{P+2,Q+2}\left[w_2 x^{\sigma_2}\left|\begin{matrix}(a_j,\alpha_j;A_j)_{1,N},(a_j,\alpha_j)_{N+1,P} \\ \left(1-\lambda-\beta+\eta-\rho R-\sigma_1 S_{p,\upsilon},\sigma_2;1\right)\end{matrix}\right.\right.$$

$$\left.\begin{matrix}\left(1-\lambda-\beta-n-\rho R-\sigma_1 S_{p,\upsilon},\sigma_2;1\right)\left(1-\lambda+\alpha+\eta-\rho R-\sigma_1 S_{p,\upsilon},\sigma_2;1\right) \\ (b_j,\beta_j)_{1,M},(b_j,\beta_j;B_j)_{M+1,Q}\left(1-\lambda-\beta-\rho R-\sigma_1 S_{p,\upsilon},\sigma_2;1\right)\end{matrix}\right] \qquad (6.4)$$

which implies the formula (6.2) in view of n times differentiation and the relation $n\Gamma(n)=\Gamma(n+1)$.

Corollary 4.1. Let $\alpha \in \mathbf{C}$ with Re (α) > 0 and let the constants $\lambda \in \mathbf{C}$, $\sigma_1, \sigma_2 > 0$ satisfy and

$$\sigma_1 \max_{1\le j\le N_1}\left[\frac{\operatorname{Re}(e_j)-1}{E_j}\right]+\sigma_2 \max_{1\le j\le N}\left[\frac{\operatorname{Re}(a_j)-1}{\alpha_j}\right] + \operatorname{Re}(\lambda)+\operatorname{Re}(\alpha)<0 \qquad (6.5)$$

Then the Riemann-Liouville fractional derivative D_-^{α} of the product of $\overline{H}$ -functions with $S_V^U[\delta t^{\rho}]$ exists and the following relation holds:

$$\left(D_-^{\alpha} t^{\lambda-1}\overline{H}^{M_1,N_1}_{P_1,Q_1}\left[w_1 t^{\sigma_1}\right]\overline{H}^{M,N}_{P,Q}\left[w_2 t^{\sigma_2}\right]S_V^U[\delta t^{\rho}]\right)(x)$$

$$=(-1)^{[\operatorname{Re}(\alpha)+1]}\, x^{\lambda+\beta-1}\sum_{R=0}^{[V/U]}\frac{(-V)_{UR}}{R!}A_{(V,R)}\left(\delta x^{\rho}\right)^{R}\sum_{\upsilon=1}^{M_1}\sum_{p=0}^{\infty}\overline{\theta}\left(S_{p,\upsilon}\right)$$

$$\left(w_1 x^{\sigma_1}\right)^{S_{p,\upsilon}}\overline{H}^{M+1,N}_{P+1,Q+1}\left[w_2 x^{\sigma_2}\left|\begin{matrix}(a_j,\alpha_j;A_j)_{1,N},(a_j,\alpha_j)_{N+1,P} \\ \left(1-\lambda+\alpha-\rho R-\sigma_1 S_{p,\upsilon},\sigma_2;1\right)\end{matrix}\right.\right.$$

$$\left.\begin{matrix}\left(1-\lambda-\rho R-\sigma_1 S_{p,\upsilon},\sigma_2;1\right) \\ \left(1-\lambda-\beta-\rho R-\sigma_1 S_{p,\upsilon},\sigma_2;1\right)\end{matrix}\right] \qquad (6.6)$$

Special Cases:

1.If in theorem 1, we reduce S_V^U to $L_V^{\alpha}(x)$ Laguerre polynomial, $\overline{H}^{M_1,N_1}_{P_1,Q_1}$ to generalized Wright hypergeometric function ${}_{P_1}\overline{\psi}_{Q_1}$ and $\overline{H}^{M,N}_{P,Q}$ to generalized Wright
Bessel function with the help of results [20, p.101, Eq. (5.1.6)], [6, p.271, Eq. (7)], [18, p.271, Eq. (9)] respectively, we get the following theorem:

$$\left(I_{0+}^{\alpha,\beta,\eta} t^{\lambda-1}\,{}_{P_1}\overline{\psi}_{Q_1}\left[w_1 t^{\sigma_1}\left|\begin{matrix}(1-e_j,E_j;\varepsilon_j)_{1,P_1} \\ (1-f_j,F_j;\Im_j)_{M_1+1,Q_1}\end{matrix}\right.\right]\overline{J}_{\zeta}^{\upsilon,\mu}\left[w_2 t^{\sigma_2}\right]L_V^{(\alpha')}[\delta t^{\rho}]\right)(x)$$

$$= x^{\lambda-\beta-1}\sum_{R=0}^{[V/U]}\frac{(-V)_{UR}}{R!}\binom{V+\alpha}{V}\frac{\left(\delta x^{\rho}\right)^{R}}{(1+\alpha)_R}$$

$$\sum_{p=0}^{\infty}\frac{\prod_{j=1}^{P_1}\left\{\Gamma\left(1-e_j+E_j\xi\right)\right\}^{\varepsilon_j}}{\prod_{j=1}^{Q_1}\left\{\Gamma\left(1-f_j+F_j\xi\right)\right\}^{\Im_j}}\left(w_1 x^{\sigma_1}\right)^{p}\overline{H}^{1,2}_{2,4}\left[w_2 x^{\sigma_2}\left|\begin{matrix}\left(1-\lambda-\rho R-\sigma_1 p,\sigma_2;1\right) \\ (0,1),(-\zeta,\upsilon;\mu)\left(1-\lambda+\beta-\rho R-\sigma_1 p,\sigma_2;1\right)\end{matrix}\right.\begin{matrix}\left(1-\lambda-\eta+\beta-\rho R-\sigma_1 p,\sigma_2;1\right) \\ \left(1-\lambda-\alpha-\eta-\rho R-\sigma_1 p,\sigma_2;1\right)\end{matrix}\right]$$

2. Once again in theorem 1, if we reduce S_V^U polynomial to $y_V\left(-\beta' x, \alpha', \beta'\right)$ Bessel polynomial, $\overline{H}_{P_1,Q_1}^{M_1,N_1}$ to generalized Riemann Zeta function $\phi(x)$ and $\overline{H}_{P,Q}^{M,N}$ to generalized hypergeometric function ${}_P\overline{F}_Q$ using results [9, p.108, Eq. (34)], [3, p.271, Eq. (1)], [6, p.471, Eq. (9)] respectively, it takes the following interesting form:

$$\Bigg(I_{0+}^{\alpha,\beta,\eta} t^{\lambda-1}\phi\left(w_1 t^{\sigma_1}, k, r\right) {}_P\overline{F}_Q\left[w_2 t^{\sigma_2}\left|\begin{matrix}(1-a_j, A_j)_{1,P}\\(1-b_j, B_j)_{1,Q}\end{matrix}\right.\right]$$

$$y_V[-\beta'\delta t^{\rho}, \alpha', \beta']\Bigg)(x) = x^{\lambda-\beta-1}\sum_{R=0}^{[V/U]}\frac{(-V)_{UR}}{R!}\left(\alpha'+V-1\right)_R\left(\delta x^{\rho}\right)^R$$

$$\sum_{p=0}^{\infty}\frac{\prod_{j=1}^{Q}\left\{\Gamma\left(1-b_j\right)\right\}^{B_j}}{\prod_{j=1}^{P}\left\{\Gamma\left(1-a_j\right)\right\}^{A_j}}\frac{\left(w_1 x^{\sigma_1}\right)^p}{\left(p+r\right)^k}\overline{H}_{P+2,Q+3}^{1,P+2}\left[w_2 x^{\sigma_2}\left|\begin{matrix}(1-a_j, A_j)_{1,P}\\(0,1)_{1,M}, (1-b_j, B_j)_{1,Q}\end{matrix}\right.\right.$$

$$\left.\begin{matrix}\left(1-\lambda-\rho R-\sigma_1 p, \sigma_2; 1\right)\left(1-\lambda-\eta+\beta-\rho R-\sigma_1 p, \sigma_2; 1\right)\\\left(1-\lambda+\beta-\rho R-\sigma_1 p, \sigma_2; 1\right)\left(1-\lambda-\alpha-\eta-\rho R-\sigma_1 p, \sigma_2; 1\right)\end{matrix}\right]$$

3. If in Theorem 2, we reduce S_V^U polynomial to Hermite polynomial $H_V(x)$, $\overline{H}_{P_1,Q_1}^{M_1,N_1}$ to H- function and $\overline{H}_{P,Q}^{M,N}$ to g_1 function with the help of [20, p.106, Eq. (5.5.4)], [8, p.4125, Eq. (20)] we arrive at the following result after a little simplification:

$$\Bigg(I_{-}^{\alpha,\beta,\eta} t^{\lambda-1}\overline{H}_{P_1,Q_1}^{M_1,N_1}\left[w_1 t^{\sigma_1}\right] g\left[r, \mu, \tau, m, w_2 t^{\sigma_2}\right]$$

$$[\delta t^{\rho}]^{V/2} H_V\left[\frac{1}{2\sqrt{\delta t^{\rho}}}\right]\Bigg)(x) = x^{\lambda-\beta-1}\sum_{R=0}^{[V/U]}\frac{(-V)_{2R}}{R!}(-1)^R\left(\delta x^{\rho}\right)^R$$

$$\sum_{\upsilon=1}^{M_1}\sum_{p=0}^{\infty}\bar{\theta}\left(S_{P,\upsilon}\right)\left(w_1 x^{\sigma_1}\right)^{S_{P,\upsilon}}\frac{\Gamma(m+1)\Gamma\left(\frac{1+\tau}{2}\right)}{\pi^{d/2}2^{m+d}\Gamma\left(\frac{d-1}{2}\right)\Gamma(r)\Gamma\left(r-\frac{\tau}{2}\right)}$$

$$\overline{H}_{5,5}^{3,3}\left[w_2 x^{\sigma_2}\left|\begin{matrix}(1-r,1;1),\left(1-r+\frac{\tau}{2},1;1\right)\\(0,1),\left(-\frac{\tau}{2},1;1\right)\left(-\mu,1;1+m\right)\end{matrix}\right.\right.$$

$$\left.\begin{matrix}\left(1-\lambda-\rho R-\sigma_1 S_{p,\upsilon}, \sigma_2\right)\left(1-\lambda+\alpha+\beta+\eta-\rho R-\sigma_1 S_{p,\upsilon}, \sigma_2\right)\\\left(1-\lambda+\beta-\rho R-\sigma_1 S_{p,\upsilon}, \sigma_2\right)\left(1-\lambda+\eta-\rho R-\sigma_1 S_{p,\upsilon}, \sigma_2\right)\end{matrix}\right]$$

where

$$\theta(S_{p,\upsilon}) = \frac{\prod_{j=1, j\neq\upsilon}^{M_1} \Gamma(b_j - \beta_j S_{p,\upsilon}) \prod_{j=1}^{N_1} \Gamma(1-a_j+\alpha_j S_{p,\upsilon})(-1)^p}{\prod_{j=M+1}^{Q_1} \Gamma(1-b_j+\beta_j S_{p,\upsilon})\} \prod_{j=N+1}^{P_1} \Gamma(a_j-\alpha_j S_{p,\upsilon}) p! \beta_\upsilon} \text{ and } S_{p,\upsilon} = \frac{b_\upsilon + p}{\beta_\upsilon}$$

The results obtained by M. Saigo and A.A. Kilbas in [14] can be easily deduced from our results. If in theorem 1 and 2 we put $w=1$, reduce the polynomial S_V^U and $\overline{H}_{P_1,Q_1}^{M_1,N_1}$ to unity and $\overline{H}_{P,Q}^{M,N}$ to familiar H-function we arrive at known results

recorded in [10, pp. 109-110, Eqs. (3.130), (3.131)]. Further, if in corollary (1.1) and (1.2) we reduce S_V^U and $\overline{H}_{P_1,Q_1}^{M_1,N_1}$ to unity, we get the results given by Srivastava H.M. [19, p. 97, Eqs. (2.4), (2.5)]. Also by reducing $\overline{H}_{P_1,Q_1}^{M_1,N_1}$ to unity, we at once get the results obtained by Chaurasia et al. [2].

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