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SOLUTION OF HEAT EQUATION BY FOURIER-BESSEL TRANSFORM

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ABSTRACT

In this paper we have discussed certain boundary value problem of heat the cylindrical shell solve by fourier-bessel transform and also discussed temperature distribution in a cylindrical shell with heat source inside the cylinder. Measuring and finding the distribution and variation is one of the significant purposes of presenting different methods for solving heat equation.

Keywords: Integral transforms, fourier-bessel transform and heat equation.

[1] Introduction:

In this article, the heat equation problem of a sector of a finite hollow cylinder is studied as an exact solution approach. The governing equations are in the form of non-homogeneous partial differential equation (PDE) with non-homogeneous boundary conditions. In order to solve the PDE equation, generalized fourier-bessel, periodic Fourier, Fourier and Laplace transforms are applied. The results are shown that this approach is suitable and systematic for solving heat equation in cylindrical coordinate.

Measuring and finding the distribution and variation of heat equation directly is not an easy work to do and not possible for some cases and that is one of the significant purposes of presenting different methods for solving heat equation. (Hoshan[1]) presented a triple integral equation method for solving heat equation . A new kind of triple integral was employed to find a solution of non-stationary heat equation in an axis-symmetric cylindrical coordinates under mixed boundary of the first and second kind conditions. (Kayhani et al[2]). introduced a general analytical solution for heat equation in cylindrical multilayer composite laminates . In the article, the direction of fiber was able to change between the layers. The boundary condition was considered linear and the method was introduced suitable for boundary conditions consisting of conduction, convection and radiation. (Cossali [3]) expressed an analytical solution of the steady periodic heat equation in a solid homogenous finite cylinder via Fourier transform, with the sole restriction of uniformity on the lateral surface and radial symmetry on the bases . A harmonic heating, as an example, introduced with simulation results.(Matysiak et al.[4]) expressed the problem of transient heat equation in a from-time-to-time arranged in layers consisting of a large number of interchanging concentric cylinders . The cylinders have a great quantity of circular homogenous isotropic rigid sectors. (Sommers and Jacobi[5]) presented an exact solution to steady heat equation in a two-dimensional hollow on a one-dimensional fin .

The fin efficiency of a high thermal conductivity was discussed. The exact solution was obtained by separation of variables method. (Jabbari et al.[6]) explained an analytical solution to a problem of one-dimensional moving heat source in a hollow FGM cylinder . Mechanical and thermal stresses were considered and the material in the case, varied continuously across the thickness. The method of solution was direct and used Bessel function. (Atefi and Talaee[7]) The separation of variable method was applied for solving the time-independent boundary condition and the Duhamel integral was used to apply for time dependent part.

In this paper, in order to solve the problem as an exact method, generalized fourier-bessel transform is used. fourier-bessel is a transformation for solving problems consisting of cylindrical coordinates, but not the hollow one.(Eldabe et al.[8]) introduced an extension of the fourier-bessel transform which was capable of solving problems in hollow cylindrical coordinates, heat equation or wave with mixed boundary values .(Povstenko[9]) expressed the radial heat in a cylinder via Laplace and fourier-bessel transform . (Akhtar[10]) presented exact solutions for rotational flow of a generalized Maxwell fluid between two circular cylinders .

In order to find the exact solution Laplace and fourier-bessel transforms were employed. (Fetecau et al.[11]) introduced exact solutions for the flow of a viscoelastic fluid induced by a circular cylinder subject to a time dependent shear stress via fourier-bessel transform . (Yu et al.[12]) expressed general temperature computational method of linear heat equation for multilayer cylinder .

[2] Theorem

Let us consider a cylinder of radial $p, q\ and\ \ height\ \ s$ and symmetrical along $x - axis$, having a heat source inside which leads axially symmetrical temperature distribution. Let (r, θ, x) be the cylindrical coordinate system and the heat equation symmetrically with respect to $x - axis$. The temperature function τ is the function of space and time.

The heat equation is given as

$$\varrho\lambda\tau_t = \mu\nabla^2\varphi + \phi(r, x, t, \tau) \qquad (1)$$

Where $\phi(r, x, t, \tau)$ is a source function.

The use of substitutions

$$\phi(r, x, t, \tau) = \tau(r, x, t) + \zeta(t)\tau(r, x, t) \qquad (2)$$

$$\psi(r, x, t) = \tau(r, x, t)exp\left\{- \int_0^t \epsilon(y)dy\right\} \qquad (3)$$

The heat equation (1) reduces to

$$\phi_t = \mu\nabla^2\psi + \frac{\psi(r,x,t)}{\varrho} \qquad (4)$$

Where $\mu = \frac{\kappa}{\varrho\lambda}$, μ is diffusivity.

κ The thermal conductivity, ϱ the density and λ is the specific heat.

Here we take the composite cylinder of variable density and suppose

$$\varrho = \varsigma \; e^{-\alpha x} \qquad (5)$$

Where ς and α are constant.

The equation (4) reduce to

$$\kappa\left[\frac{\partial^2\psi(r,x,t)}{\partial r^2} + \frac{1}{r}\frac{\partial\psi(r,x,t)}{\partial r} + \frac{\partial^2\psi(r,x,t)}{\partial x^2}\right] + \frac{\xi(r,x,t)}{\varsigma \; e^{-x\alpha}} - \frac{\partial\psi(r,x,t)}{\partial t} = 0 \qquad (6)$$

$$u(r, x, t)|_{r=p} = \Phi_1(x, t) \qquad (7)$$

$$u(r, x, t)|_{r=q} = \Phi_2(z. t) \qquad (8)$$

$$u(r,x,t)|_{x=0} = \Phi_3(t) \qquad (9)$$

$$u(r,x,t)|_{x=s} = \Phi_4(t) \qquad (10)$$

$$u(r,x,t)|_{t=0} = \Upsilon \qquad (11)$$

Where Υ is constant.

[3] Solution

Using fourier-bessel transform between the limit $p\ to\ q$ with respect to r, given by

$$\bar{f}(\beta_i) = \int_p^q rf(r)\big(j_v(\beta_i r)y_v(a\beta_i) - y_v(\beta_i r)j_v(a\beta_i)\big)dr \qquad (12)$$

With inversion series

$$f(r) = \frac{\pi^2}{2}\sum_i \frac{\bar{f}(\beta_i){j_v}^2(b\beta_i){\beta_i}^2[j_v(\beta_i r)y_v(a\beta_i) - y_v(\beta_i r)j_v(a\beta_i)]}{{j_v}^2(a\beta_i) - {j_v}^2(b\beta_i)} \qquad (13)$$

Where summation over i extend over all the positive roots of the equation

$$j_v(\beta_i a)y_v(b\beta_i) - y_v(\beta_i a)j_v(b\beta_i) = 0$$

and there operation property is

$$\int_p^q r\left(\frac{\partial^2 f}{\partial r^2} + \frac{1}{r}\frac{\partial f}{\partial r} - \frac{v^2}{r^2}f\right)\big(j_v(\beta_i r)y_v(a\beta_i) - y_v(\beta_i r)j_v(a\beta_i)\big)dr$$

$$= \frac{2}{\pi}\frac{j_v(a\beta_i)}{j_v(b\beta_i)}f(b) - \frac{2}{\pi}f(a) - {\beta_i}^2\bar{f}(\beta_i) \qquad (14)$$

Now by equation (6), we get

$$\kappa\int_p^q r\left(\frac{\partial^2\psi}{\partial r^2} + \frac{1}{r}\frac{\partial\psi}{\partial r}\right)\big(j_0(\beta_i r)y_0(a\beta_i) - y_0(\beta_i r)j_0(a\beta_i)\big)dr + \kappa\frac{\partial^2\bar{\psi}(\beta_i,x,t)}{\partial x^2} +$$
$$\frac{\bar{\phi}(\beta_i,x,t)}{\varsigma}e^{x\alpha} - \frac{\partial\bar{\psi}(\beta_i,x,t)}{\partial t} = 0 \qquad (15)$$

Or

$$\kappa\left[\frac{2}{\pi}\frac{j_0(a\beta_i)}{j_0(b\beta_i)}\psi(q,x,t)-\frac{2}{\pi}\psi(p,x,t)-{\beta_i}^2\bar{\psi}(\beta_i,x,t)\right]+\mathrm{K}\frac{\partial^2\bar{\psi}(\beta_i,x,t)}{\partial x^2}+\frac{\bar{\phi}(\beta_i,x,t)}{\varsigma}e^{\alpha x}-\frac{\partial\bar{\psi}(\beta_i,x,t)}{\partial t}=0 \qquad (16)$$

$$\kappa\left[\frac{2}{\pi}\frac{j_0(a\beta_i)}{j_0(b\beta_i)}\phi_2(x,t)-\frac{2}{\pi}\phi_1(x,t)-{\beta_i}^2\bar{\psi}(\beta_i,x,t)\right]+\kappa\frac{\partial^2\bar{\psi}(\beta_i,x,t)}{\partial x^2}+\frac{\bar{\phi}(\beta_i,x,t)}{\varsigma}e^{\alpha x}-\frac{\partial\bar{\psi}(\beta_i,x,t)}{\partial t}=0 \qquad (17)$$

Now using finite Fourier Cosine transforms between the limit 0 $to\ s$ with respect to x on the above result, be obtain

$$\int_0^s\left[\kappa\left[\frac{2}{\pi}\frac{j_0(a\beta_i)}{j_0(b\beta_i)}\phi_2(x,t)-\frac{2}{\pi}\phi_1(x,t)-{\beta_i}^2\bar{\psi}(\beta_i,x,t)\right]+\kappa\frac{\partial^2\bar{\psi}(\beta_i,x,t)}{\partial x^2}+\frac{\bar{\phi}(\beta_i,x,t)}{\varsigma}e^{\alpha x}-\frac{\partial\bar{\psi}(\beta_i,x,t)}{\partial t}\right]\cos\frac{m\pi x}{d}dx=0$$

Or

$$\kappa\int_0^s\frac{\partial^2\bar{\psi}}{\partial x^2}\cos\frac{m\pi x}{d}dx+\int_0^s\frac{\bar{\phi}(\beta_i,x,t)}{\varsigma}e^{\alpha x}\cos\frac{m\pi x}{d}dx+\kappa\left[\frac{2}{\pi}\frac{j_0(a\beta_i)}{j_0(b\beta_i)}\overline{\phi_2}(m,t)-\frac{2}{\pi}\overline{\phi_1}(m,t)-{\beta_i}^2\bar{\bar{\psi}}(\beta_i,m,t)\right]=0 \qquad (18)$$

On using boundary conditions and operational property, we get

$$\kappa\left[\frac{m\pi}{s}(-1)^{m+1}\phi_4(t)+\phi_3(t)\right]-\kappa\left(\frac{m\pi}{s}\right)^2\bar{\bar{\psi}}(\beta_i,m,t)+\bar{G}(\beta_i,m,t)-\frac{\partial\bar{\bar{\psi}}}{\partial t}+\kappa\left[\frac{2}{\pi}\frac{j_0(a\beta_i)}{j_0(b\beta_i)}\overline{\phi_2}(m,t)-\frac{2}{\pi}\overline{\phi_1}(m,t)-{\beta_i}^2\bar{\bar{\psi}}(\beta_i,m,t)\right]=0 \qquad (19)$$

Where

$$\bar{G}(\beta_i,m,t)=\int_0^s\frac{\bar{\phi}(\beta_i,x,t)}{\varsigma}e^{\alpha x}\cos\frac{m\pi x}{d}dx \qquad (20)$$

Now apply the Laplace transform and apply inverse laplace transform on obtained result, further using inverse Fourier cosine transform:

$$\bar{\psi}(\beta_i,x,t)=\frac{2}{h}\sum_{m=1}^{\infty}\bar{\bar{\psi}}(\beta_i,m,t)\cos\frac{m\pi x}{d} \qquad (21)$$

$$\bar{u}(\beta_i, z, t) = \frac{2}{h}\sum_{m=1}^{\infty}\Bigg[\kappa\frac{m\pi}{h}(-1)^{m+1}\int_0^t \phi_4(s)e^{-\left(\left(\frac{m\pi}{h}\right)^2+\beta_i^{\,2}\right)\kappa(t-s)}ds +$$

$$\kappa\int_0^t \phi_3(s)e^{-\left(\left(\frac{m\pi}{h}\right)^2+\beta_i^{\,2}\right)\kappa(t-s)}ds + \int_0^t \bar{G}(\beta_i, m, s)e^{-\left(\left(\frac{m\pi}{h}\right)^2+\beta_i^{\,2}\right)\kappa(t-s)}ds +$$

$$\kappa\frac{2}{\pi}\frac{j_0(a\beta_i)}{j_0(b\beta_i)}\int_0^t \overline{\phi_2}(m,s)e^{-\left(\left(\frac{m\pi}{h}\right)^2+\beta_i^{\,2}\right)\kappa(t-s)}ds + \kappa\frac{2}{\pi}\int_0^t \overline{f_1}(m,s)e^{-\left(\left(\frac{m\pi}{h}\right)^2+\beta_i^{\,2}\right)\kappa(t-s)}ds +$$

$$\chi e^{-\left(\left(\frac{m\pi}{h}\right)^2+\beta_i^{\,2}\right)\kappa(t)}\Bigg]\cos\frac{m\pi x}{d} \qquad (22)$$

On using Inversion theorem of Fourier-Bessel transform given by the equation (13), we get

$$u(r, x, t) = \frac{\pi^2}{2}\sum_i\Bigg[\frac{2}{h}\sum_{m=1}^{\infty}\Bigg[\kappa\frac{m\pi}{h}(-1)^{m+1}\int_0^t \phi_4(s)e^{-\left(\left(\frac{m\pi}{h}\right)^2+\beta_i^{\,2}\right)\kappa(t-s)}ds +$$

$$\kappa\int_0^t \phi_3(s)e^{-\left(\left(\frac{m\pi}{h}\right)^2+\beta_i^{\,2}\right)\kappa(t-s)}ds + \int_0^t \bar{G}(\beta_i, m, s)e^{-\left(\left(\frac{m\pi}{h}\right)^2+\beta_i^{\,2}\right)\kappa(t-s)}ds +$$

$$\kappa\frac{2}{\pi}\frac{j_0(a\beta_i)}{j_0(b\beta_i)}\int_0^t \overline{\phi_2}(m,s)e^{-\left(\left(\frac{m\pi}{h}\right)^2+\beta_i^{\,2}\right)\kappa(t-s)}ds + \kappa\frac{2}{\pi}\int_0^t \overline{\phi_1}(m,s)e^{-\left(\left(\frac{m\pi}{h}\right)^2+\beta_i^{\,2}\right)\kappa(t-s)}ds +$$

$$\chi e^{-\left(\left(\frac{m\pi}{h}\right)^2+\beta_i^{\,2}\right)\kappa(t)}\Bigg]\cos\frac{m\pi z}{h}\Bigg]\frac{j_v^{\,2}(b\beta_i)\beta_i^{\,2}[j_v(\beta_i r)y_v(a\beta_i)-y_v(\beta_i r)j_v(a\beta_i)]}{j_v^{\,2}(a\beta_i)-j_v^{\,2}(b\beta_i)} \qquad (23)$$

This gives temperature distribution in cylindrical shell.

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