Journal of Analysis and Computation (JAC)
(An International Peer Reviewed Journal), www.ijaconline.com, ISSN 0973-2861
Volume XVI, Issue IV, Jan-June 2022

GRAPH EQUATIONS FOR LINE GRAPHS, MIDDLE GRAPHS, SEMI-SPLITTING BLOCK GRAPHS AND SEMI-LINE SPLITING BLOCK GRAPHS

Dr.Sangeeta Bhat
Assistant Professor, Department of Mathematics and statistics, K.J.Somaiya college of Arts and commerce (Autonomous),Vidyavihar,Mumbai,400077,INDIA

ABSTRACT

Graph equations are those in which the unknowns are graphs. During the past fifty years, the development in these areas has emerged as one of the significant branches in Graph Theory. Let L(G) and M(G) denote the line graph and middle graph respectively. In this paper, we solve the graph equations $L(G) \cong S_B(H)$, $\overline{L(G)} \cong S_B(H)$, $M(G) \cong S_B(H)$, $\overline{M(G)} \cong S_B(H)$, $L(G) \cong BL_S(H)$, $\overline{L(G)} \cong BL_S(H)$, $M(G) \cong BL_S(H)$, $\overline{M(G)} \cong BL_S(H)$.

The symbol ≅ stands for isomorphism between two graphs.

Keywords: Line graph, middle graph, Semi splitting block graphs ,semi line splitting block graphs.Mathematics subject classification: 05C99

[1] INTRODUCTION

We require the following definitions.

A graph G is called a block it has more than one vertex, is connected and has no cut-vertex. A block of a graph G is a maximal subgraph of G which is itself a block.

If $B = \{u_1, u_2, \ldots u_r ; r \geq 2\}$ is a block of G, then we say that vertex u_1 and block B are incident with each other, as are u_2 and B so on. Similarly, if B is a block with edge set $\{e_1, e_2, \ldots, e_r ; r \geq 1\}$, there we say that edge e_i and block B are incident with each other, $1 \leq i \leq r$.

For a graph G, let V(G), E(G) and b(G) denote its vertex set, edge set and set of blocks of G respectively.

The open-neighborhood N(u) of a vertex u in V(G) is the set of all vertices adjacent to u.

$N(u) = \{ v / uv \in E(G) \}$

For each vertex u_i of G, a new vertex u_i' is taken and the resulting set of vertices is denoted by $V_1(G)$.

The semi-splitting block graph $S_B(G)$ of a graph G is defined as the graph having vertex set $V(G) \cup V_1(G) \cup b(G)$ and two vertices are adjacent if they correspond to a adjacent vertices of G or one corresponds to a vertex u_i' of $V_1(G)$ and the other to vertex w_j of G and w_j is in $N(u_i)$ or one corresponds to a vertex u_i of V(G) and the other to a vertex b_i of b(G) and u_i is in b_i (see [7]).

The open-neighborhood $N(e_i)$ of an edge e_i in E(G) is the set of edges adjacent to e_i.

$$N(e_i) = \{ e_j / e_i \text{ and } e_j \text{ are adjacent in G} \}.$$

For each edge e_i of G, a new vertex e_i' is taken and resulting set of vertices is denoted by $E_1(G)$.

For a graph G, we define the semi-line splitting block graph $BL_S(G)$ of a graph G as the graph having vertex set $E(G) \cup E_1(G) \cup b(G)$ with two vertices are adjacent if they correspond to adjacent edges of G or one corresponds to an element e_i' of $E_1(G)$ and the other to an element e_j of E(G) and e_j is in $N(e_i)$ or one corresponds to an edge e_i of E(G) and the other to a vertex b_i of b(G) and e_i lies on b_i.

A graph G, its semi-splitting block graph $S_B(G)$ and semi-line splitting block graph $BL_s(G)$ are shown in Figure 1.

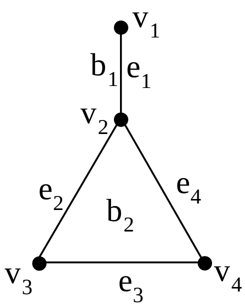
Sastry and Syam Prasad Raju [8] solved the following graph equations:

$$L(G) \cong P(H), \quad \overline{L(G)} \cong P(H)$$

$$M(G) \cong P(H), \quad \overline{M(G)} \cong P(H)$$

$$P(G) \cong T(H), \quad \overline{P(G)} \cong T(H).$$

G :



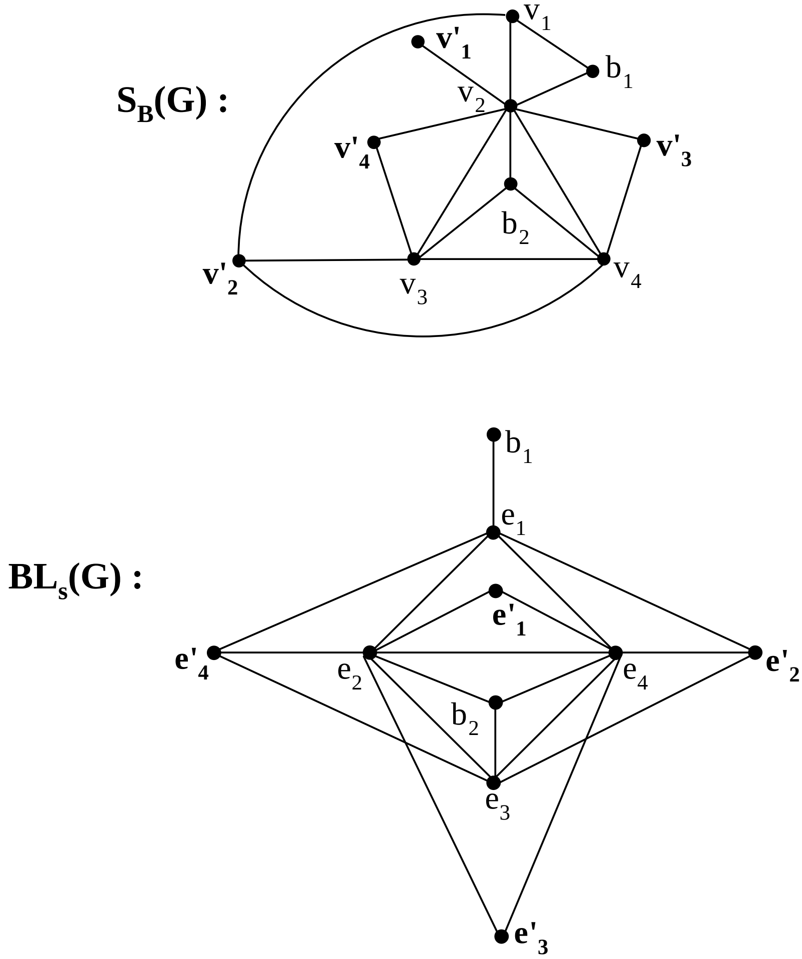


Figure 1.

In [2], Akka solved graph equations

$L(G) \cong e(H)$, $L(G) \cong e_v(H)$ and

$L(G) \cong e_e(H)$.

Basavanagoud and Mathad [3] solved the following graph equations.

$L(G) \cong S(H)$, $J(G) \cong S(H)$,

$M(G) \cong S(H)$, $\overline{M(G)} \cong S(H)$

$L(G) \cong L_S(H)$, $J(G) \cong L_S(H)$

$M(G) \cong L_S(H)$, $\overline{M(G)} \cong L_S(H)$

In [5], Cvetkovič and Simič obtained a bibliography of graph equations.

In this paper, we solve the following graph equations:

$L(G) \cong S_B(H)$ $\overline{L(G)} \cong S_B(H)$

$M(G) \cong S_B(H)$ $\overline{M(G)} \cong S_B(H)$

$L(G) \cong BL_S(H)$ $\overline{L(G)} \cong BL_S(H)$

$M(G) \cong BL_S(H)$ $\overline{M(G)} \cong BL_S(H)$.

Beineke has shown in [4] that a graph G is a line graph if and only if G has none of the nine specified graphs F_i, i = 1, 2, …, 9 as an induced subgraph . We depict here three of the nine graphs which are useful to extract our later results. And also $\overline{F_i}$, i = 1, 2, …, 9 is the complement of F_i . We depict here three of the nine graphs which are useful to extract our later results. These are $F_1 = K_{1,3}$, F_5 and $F_3 = K_5 – x$, where x is any edge of K_5.

A graph G^+ is the endedge graph of a graph G if G^+ is obtained from G by adjoining an endedge $u_i u_i'$ at each vertex u_i of G (see [1]). Hamada and Yoshimura have proved in [6] that $M(G) \cong L(G^+)$.

2. THE SOLUTION OF $L(G) \cong S_B(H)$

Any graph H which is a solution of the above equation, satisfies the following properties:

i) H must be a line graph, since H is an induced subgraph of $S_B(H)$.

ii) H does not contain a cut-vertex, since otherwise, F_1 would be an induced subgraph of $S_B(H)$.

iii) H does not contain a cycle C_n, $n \geq 3$, since otherwise, F_1 is an induced subgraph of $S_B(H)$.

There are two distinct cases to consider.

Case 1. Suppose H is connected. Then H is K_1 or K_2. The corresponding G is $2K_1^+$ or $K_{1,2}^+$ respectively.

Case 2. Suppose H is disconnected. Then H is nK_1, $n \geq 2$; nK_2, $n \geq 2$ or $nK_1 \cup nK_2$, n, m $\geq$ 1.

For $H = nK_1$, $n \geq 2$, and $G = 2nK_1^+$

For $H = nK_2$, $n \geq 2$, and $G = nK_{1,2}^+$

For $H = nK_1 \cup mK_2$, n, m $\geq$ 1, and $G = 2nK_1^+ \cup mK_{1,2}^+$.

From the above discussion, we conclude the following:

THEOREM 2.1 The following pairs (G, H) are all pairs of graphs satisfying the graph equation $L(G) \cong S_B(H)$:

$(2nK_1^+, nK_1), n \geq 1; (nK_{1,2}^+, nK_2), n \geq 1;$ and

$(2nK_1^+ \cup mK_{1,2}^+, nK_1 \cup mK_2)$, n, m $\geq$ 1.

3. THE SOLUTION OF $\overline{L(G)} \cong S_B(H)$

In this case, H satisfies the following properties:

i) If H has atleast one edge, then it is connected, since otherwise, $\overline{F_1}$ is an induced subgraph of $S_B(H)$.

ii) H does not contain a cut-vertex, since otherwise, $\overline{F_3}$ is an induced subgraph of $S_B(H)$.

iii) H does not contain two independent vertices which lie on the same block, since otherwise, $\overline{F_5}$ is an induced subgraph of $S_B(H)$.

From observations (i), (ii) and (iii), we obtain that H is nK_1, $n \geq 1$ or a block. We consider two distinct cases:

Case 1. If $H = nK_1$, $n \geq 1$, then $G = K_{1,2n}$.

Case 2. If H is a block, then it follows from observation (iii), that $H = K_n$, $n \geq 2$. The corresponding G is $K^+_{1,n}$.

In this way the graph equation is solved and we obtain the following.

THEOREM 3.1. The following pairs (G, H) are all pairs of graphs satisfying the graph equation $\overline{L(G)} \cong S_B(H)$:

$(K_{1,2n} , nK_1)$, $n \geq 1$; and $(K^+_{1,n} , K_n)$, $n \geq 2$.

4. THE SOLUTION OF $M(G) \cong S_B(H)$

Theorem 2.1 provides the solutions of the graph equation $L(G) \cong S_B(H)$. All these solutions are of the form (G^+, H). Therefore the solutions of the equation 3 are $(2nK_1 , nK_1)$, $n \geq 1$; $(nK_{1,2} , nK_2)$, $n \geq 1$; and $(2nK_1 \cup mK_{1,2} , nK_1 \cup mK_2)$, $n, m \geq 1$.

Now, we state the following result.

THEOREM 4.1. The solutions (G, H) of the graph equation $M(G) \cong S_B(H)$ are $(2nK_1 , nK_1)$, $n \geq 1$; $(nK_{1,2} , nK_2)$, $n \geq 1$; and $(2nK_1 \cup mK_{1,2} , nK_1 \cup mK_2)$, $n, m \geq 1$.

5. THE SOLUTION OF $\overline{M(G)} \cong S_B(H)$

Theorem 3.1. gives solution of the equation $\overline{L(G)} \cong S_B(H)$. Among these solutions $(K^+_{1,n} , K_n)$, $n \geq 2$ is of the form (G^+, H). Therefore, the solution of the equation $\overline{M(G)} \cong S_B(H)$ is $(K_{1,n} , K_n)$, $n \geq 2$.

Thus we obtain the following result.

THEOREM 5.1. There is only one solution $(K_{1,n} , K_n)$, $n \geq 2$ of the graph equation $\overline{M(G)} \cong S_B(H)$.

6. THE SOLUTION OF $L(G) \cong BL_S(H)$

Any graph H which is a solution of the above equation satisfies the following properties:

i) H does not contain a cut-vertex, since otherwise, F_1 is an induced subgraph of $BL_S(H)$.

ii) H does not contain a cycle C_n, $n \geq 3$, since otherwise, F_1 is an induced subgraph of BL_S (H).

From observations (i) and (ii), it implies that $\Delta(H) = 1$. Therefore H is nK_2, $n \geq 1$. The corresponding G is $(n(K_{1,2} \cup K_2), nK_2)$, $n \geq 1$.

Thus we have the following result:

THEOREM 6.1. The solutions (G, H) of the graph equation $L(G) \cong BL_S(H)$ are $(n(K_{1,2} \cup K_2), nK_2)$, $n \geq 1$.

7. THE SOLUTION OF $\overline{L(G)} \cong BL_S(H)$

We first observe that H satisfies the following properties:

i) If H has atleast one edge, then it is connected, since otherwise, $\overline{F_3}$ is an induced subgraph of $BL_S(H)$.

ii) H does not contain two independent edges, since otherwise, $\overline{F_5}$ is an induced subgraph of $BL_S(H)$.

iii) H does not contain more than one cut-vertex, since otherwise, $\overline{F_3}$ is an induced subgraph of $BL_S(H)$.

iv) H does not contain cut-vertex which lies on blocks other than K_2, since otherwise, $\overline{F_3}$ is an induced subgraph of $BL_S(H)$.

v) H does not contain $K_{1,3}$ as an induced subgraph, since otherwise, $\overline{F_1}$ is an induced subgraph of $BL_S(H)$.

There are two distinct cases to consider.

Case 1. Suppose H has cut-vertex. Then H is $K_{1,2}$ and corresponding G is $K_3 \bullet K_3$.

Case 2. Suppose H has no cut-vertices. Then H is K_2 or K_3. The corresponding G is K_2^+ or $K_{1,3}^+$ respectively.

Thus we obtain the following result.

THEOREM 7.1. The following pairs (G, H) are all pairs of graphs satisfying the graph equation $\overline{L(G)} \cong BL_S(H)$:

$(K_3 \bullet K_3,\ K_{1,2})$; (K_2^+, K_2); and $(K_{1,3}^+, K_3)$.

8. THE SOLUTION OF $M(G) \cong BL_S(H)$

Theorem 6.1 gives solutions for the equation $L(G) \cong BL_S(H)$. But none of these is of the form (G^+, H). Hence, there is no solution of the equation $M(G) \cong BL_S(H)$.

Now, we state the following result.

THEOREM 8.1. There is no solution of the graph equation $M(G) \cong BL_S(H)$.

9. THE SOLUTION OF $\overline{M(G)} \cong BL_s(H)$

Theorem 7.1 gives three pair of graphs ($K_3 \bullet K_3$, $K_{1,2}$); (K_2^+, K_2); and ($K_{1,3}^+$, K_3) which are the solutions of the equation $\overline{L(G)} \cong BL_S(H)$. Among these pairs, ($K_2^+$, K_2) and ($K_{1,3}^+$, K_3) are of the form (G^+, H). Therefore, the solution of the equation $\overline{M(G)} \cong BL_S(H)$ are (K_2, K_2) and ($K_{1,3}$, K_3).

Now, we state the following result.

THEOREM 9.1. The solutions (G,H) of the graph equation $\overline{M(G)} \cong BL_S(H)$ are (K_2, K_2); and ($K_{1,3}$, K_3).

REFERENCES

[1]. T. Akiyama, T. Hamada and I. Yoshimura, Graph equations for line graphs, total graphs and middle graphs.TRU Math. 12 (2) (1976).

[2]. D. G. Akka, Graph equations for line graphs, entire graphs and semientire graphs. J. of Math. and Phy. Sci., 19(6) (1985), 511-518.

[3]. B. Basavanagoud and Veena N. Mathad, Graph equations for line graphs, jump graphs, middle graphs, splitting graphs and line splitting graphs. Preprint.

[4]. L. W. Beineke, Derived graphs and digraphs, Beiträge zur Graphentheorie, (Manebach 1967), 17-24

[5]. D. M. Cvetković and S. K. Simić, , A bibliography of graph equations, J. of Graph Theory, 3 (1979), 311-324.

[6]. T Hamada and I. Yoshimura, Traversability and connectivity of the middle graph of a graph. Discrete Math. 14 (1976), 247-256
.

[7]. V. R. Kulli and K. M. Niranjan, The semi-splitting block graph of a graph. Preprint

[8]. D. V. S. Sastry and B. Syam Prasad Raju, Graph equations for line graphs, total graphs, middle graphs and quasi-total graphs, Discrete Math., 48(1984), 113-119.