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ZAGREB INDICES AND F-INDEX OF NEW TENSOR PRODUCTS OF GENERALIZED MIDDLE GRAPHS

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ABSTRACT

For a molecular graph G, the first Zagreb index is equal to the sum of squares of degree of all vertices, the second Zagreb index is equal to the sum of products of the degrees of pairs of adjacent vertices and the F-index or forgotten topological index is defined as the sum of cubes of degree of all its vertices. In this paper, we introduce seven new tensor products of generalized middle graphs and investigate first and second Zagreb indices, F-indices, coindices of these new graphs and their complements.

Keywords - Tensor product, Zagreb indices, Zagreb coindices, F-index, generalized middle graphs.

[1] INTRODUCTION

The topological indices are graph invariants which are numerical values associated with molecular graphs. In mathematical chemistry, these are known as molecular descriptors. The topological indices play vital role in mathematical chemistry, especially quantitative structure-property relationship (QSPR) and quantitative structure-activity relationship (QSAR) investigations. For chemical applications of topological indices refer [18, 29].

In this paper, we are concerned with simple graphs, having no directed or weighted edges and no loops. Let G be a finite graph with n vertices and m edges is called (n, m) graph. We denote vertex set and edge set of graph G as $V(G)$ and $E(G)$, respectively. The *complement of a graph* G is denoted by $\overline{G}$ and is defined as the graph whose vertex set is $V(G)$ in which two vertices are adjacent if and only if they are not adjacent in G. Obviously, $\overline{G}$ has n vertices and $\binom{n}{2} - m$ edges. The *neighbourhood* of a vertex $u \in V(G)$ is defined as the set $N_G(u)$ consisting of all vertices v which are adjacent to u in G. The *degree* of a vertex $u \in V(G)$, denoted by $d_G(u)$ and is equal to $|N_G(u)|$. The line graph $L(G)$ of a graph G is the graph with vertex set $E(G)$ and two vertices are adjacent in $L(G)$ if and only if the corresponding edges in G are adjacent. The line graph $L(G)$ has order $n_L = m$ and size

$m_L = -m + \frac{1}{2}\sum_{i=1}^{n} d_G(v_i)^2$. The subdivision graph $S(G)$ of a graph G whose vertex set is $V(G) \cup E(G)$ where two vertices are adjacent if and only if one is a vertex of G and other is an edge of G incident with it. The partial complement of subdivision graph $S^*(G)$ of a graph G whose vertex set is $V(G) \cup E(G)$ where two vertices are adjacent if and only if one is a vertex of G and other is an edge of G nonincident with it. For undefined graph theoretic terminologies and notations refer [22].

For a molecular graph G, *first Zagreb index* was defined by Gutman and Trinajstić [20] in 1972 as

$$M_1(G) = \sum_{v \in V(G)} d_G(v)^2.$$

The second Zagreb index was defined in [19] as

$$M_2(G) = \sum_{uv \in E(G)} d_G(u) \cdot d_G(v).$$

The first Zagreb index [27] can also be expressed as

$$M_1(G) = \sum_{uv \in E(G)} (d_G(u) + d_G(v)).$$

The first and second Zagreb coindices [1] were defined respectively as

$$\overline{M_1}(G) = \sum_{uv \notin E(G)} (d_G(u) + d_G(v)),$$

and

$$\overline{M_2}(G) = \sum_{uv \notin E(G)} (d_G(u) \cdot d_G(v)).$$

For basic properties of Zagreb indices refer [17, 20] and for Zagreb indices of graph operation refer [1, 12, 23, 31]. Another degree based graph invariant called *forgotten topological index* or F- index was put forward by Furtula and Gutman [15] is defined as

$$F(G) = \sum_{v \in V(G)} d_G(v)^3 = \sum_{uv \in E(G)} (d_G(u)^2 + d_G(v)^2).$$

Its coindex [14] is given by

$$\overline{F}(G) = \sum_{uv \notin E(G)} (d_G(u)^2 + d_G(v)^2).$$

See [15] for basic properties and [2, 14] for F-index of graph operations.
The *hyper Zagreb index* was introduced by Shirdel et al., in [28] which is defined as

$$HM(G) = \sum_{uv \in E(G)} (d_G(u) + d_G(v))^2$$

and hyper Zagreb coindex was introduced by Veylanki et al., in [30] as

$$\overline{HM}(G) = \sum_{uv \notin E(G)} (d_G(u) + d_G(v))^2.$$

For basic properties of hyper Zagreb index and coindex refer [16] and for graph operations refer [3, 5, 28, 30].

The Zagreb indices interms of edge-degrees reformulated by Milic$'$evic$'$ et al. in 2004 [26] . The first reformulated Zagreb index is defined by,

$$EM_1(G) = \sum_{e \in E(G)} d_G(e)^2.$$

The forgotten topological indices reformulated interms of edge-degrees. The reformulated F-index [24] is defined by,

$$EF(G) = \sum_{e \in E(G)} d_G(e)^3.$$

For a graph $G = (V, E)$, let G^0 be the graph with $V(G^0) = V(G)$ and with no edges, G^1 be the graph with $V(G^1) = V(G)$, $G^+ = G$ and $G^- = \overline{G}$.

Definition 1 *[11] Given graph $G = (V, E)$ and three variables $x, y, z \in \{0,1,+,-\}$ the xyz-transformation T^{xyz} of G is the graph with vertex set $V(T^{xyz}) = V \cup E$ and the edge set $E(T^{xyz}) = E(G^x) \cup E(L(G)^y) \cup E(W)$, where $W = S(G)$ if $z = +$, $W = S^*(G)$ if $z = -$, W is graph with $V(W) = V \cup E$ and with no edges if $z = 0$, and W is the complete bipartite graph with parts V and E if $z = 1$.*

The generalized middle graph G^{xy}, introduced by Basavanagoud et al. [4], is a graph having $V(G) \cup E(G)$ as the vertex set and $\alpha, \beta \in V(G) \cup E(G)$, α and β are adjacent in G^{xy} if and only if one of the following conditions holds:

1. $\alpha, \beta \in E(G)$, α, β are adjacent in G if $x = +$ and α, β are not adjacent in G if $x = -$.
2. $\alpha \in V(G)$ and $\beta \in E(G)$, α, β are incident in G if $y = +$ and α, β are not incident in G if $y = -$.

The generalized middle graphs and their complements are shown in Figure 1.

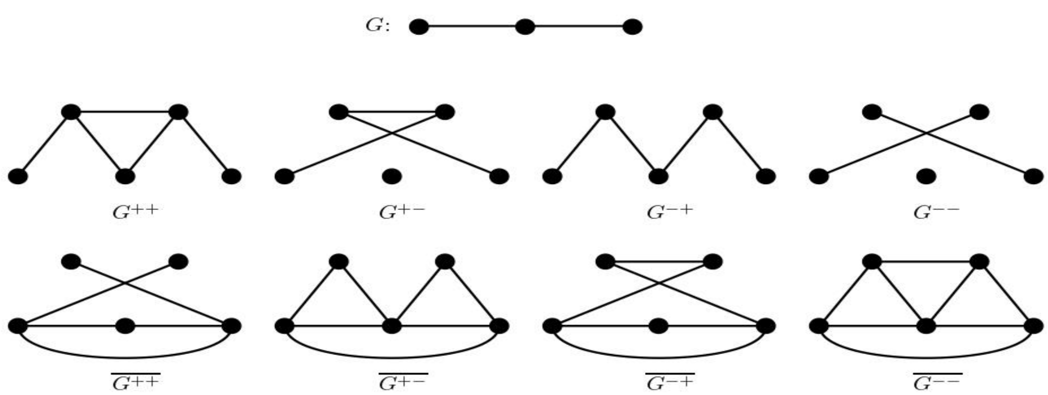

Figure 1: The generalized middle graphs and their complements.

In the recent paper [13], Eliasi and Taeri introduced four new operations on graphs as follows:

Definition 2 *[13] Let $F \in \{S, T_2, T_1, T\}$. The F-sums of G_1 and G_2, denoted by $G_1 +_F G_2$, is a graph with the set of vertices $V(G_1 +_F G_2) = (V(G_1) \cup E(G_1)) \times V(G_2)$ and two vertices (u_1, u_2) and (v_1, v_2) of $G_1 +_F G_2$ are adjacent if and only if $[u_1 = v_1 \in V(G_1)$ and*

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$u_2v_2 \in E(G_2)]$ *or* $[u_2 = v_2 \in V(G_2)$ *and* $u_1v_1 \in E(F(G_1))]$.

Thus, authors in [13] obtained four new graph operations as $G_1 +_S G_2$, $G_1 +_{T_2} G_2$, $G_1 +_{T_1} G_2$ and $G_1 +_T G_2$ and studied the Wiener indices of these graphs. In [12], Deng et al. gave the expressions for first and second Zagreb indices of these new graphs.

The *tensor product* [25] $G_1 \otimes G_2$ of two graphs G_1 and G_2 of order n_1 and n_2 respectively, is defined as the graph with vertex set $V_1 \times V_2$ and (u_1, v_1) is adjacent with (u_2, v_2) if and only if $u_1u_2 \in E(G_1)$ and $v_1v_2 \in E(G_2)$.

Basavanagoud et al., introduced four new tensor products of graphs by extending F-sums of graphs on cartesian product to tensor product in [8] as follows:

Definition 3 *[8] Let* $\mathcal{F}$ *be the one of the symbols* S, T_2, T_1 *or* T. *The* $\mathcal{F}$ *−tensor product* $G_1 \otimes_{\mathcal{F}} G_2$ *is a graph with the set of vertices* $V(G_1 \otimes_{\mathcal{F}} G_2) = (V(G_1) \cup E(G_1)) \times V(G_2)$ *and two vertices* (u_1, u_2) *and* (v_1, v_2) *of* $G_1 \otimes_{\mathcal{F}} G_2$ *are adjacent if and only if* u_1 *is adjacent to* v_1 *in* $\mathcal{F}(G_1)$ *and* u_2 *is adjacent to* v_2 *in* G_2.

Thus, authors in [8] obtained four new graph operations as $G_1 \otimes_S G_2$, $G_1 \otimes_{T_2} G_2$, $G_1 \otimes_{T_1} G_2$ and $G_1 \otimes_T G_2$ and studied the first and second Zagreb indices of these graphs. Basavanagoud et al. [9] obtained F-index and hyper Zagreb index of $G_1 \otimes_S G_2$, $G_1 \otimes_{T_2} G_2$, $G_1 \otimes_{T_1} G_2$ and $G_1 \otimes_T G_2$.

Motivated from [8], we introduce new tensor products of generalized middle graphs by extending $\mathcal{F}$ −sums of graphs on cartesian product to tensor product as follows:

Definition 4 *Let* $\mathcal{F}$ *be the one of the symbols* G^{xy} *or* $\overline{G^{xy}}$, *where* $x, y \in \{+, -\}$. *The* $\mathcal{F}$ *−tensor product* $G_1 \otimes_{\mathcal{F}} G_2$ *is a graph with the set of vertices* $V(G_1 \otimes_{\mathcal{F}} G_2) = (V(G_1) \cup E(G_1)) \times V(G_2)$ *and two vertices* (u_1, u_2) *and* (v_1, v_2) *of* $G_1 \otimes_{\mathcal{F}} G_2$ *are adjacent if and only if* u_1 *is adjacent to* v_1 *in* $\mathcal{F}(G_1)$ *and* u_2 *is adjacent to* v_2 *in* G_2.

We illustrate this definition in Figure 2.

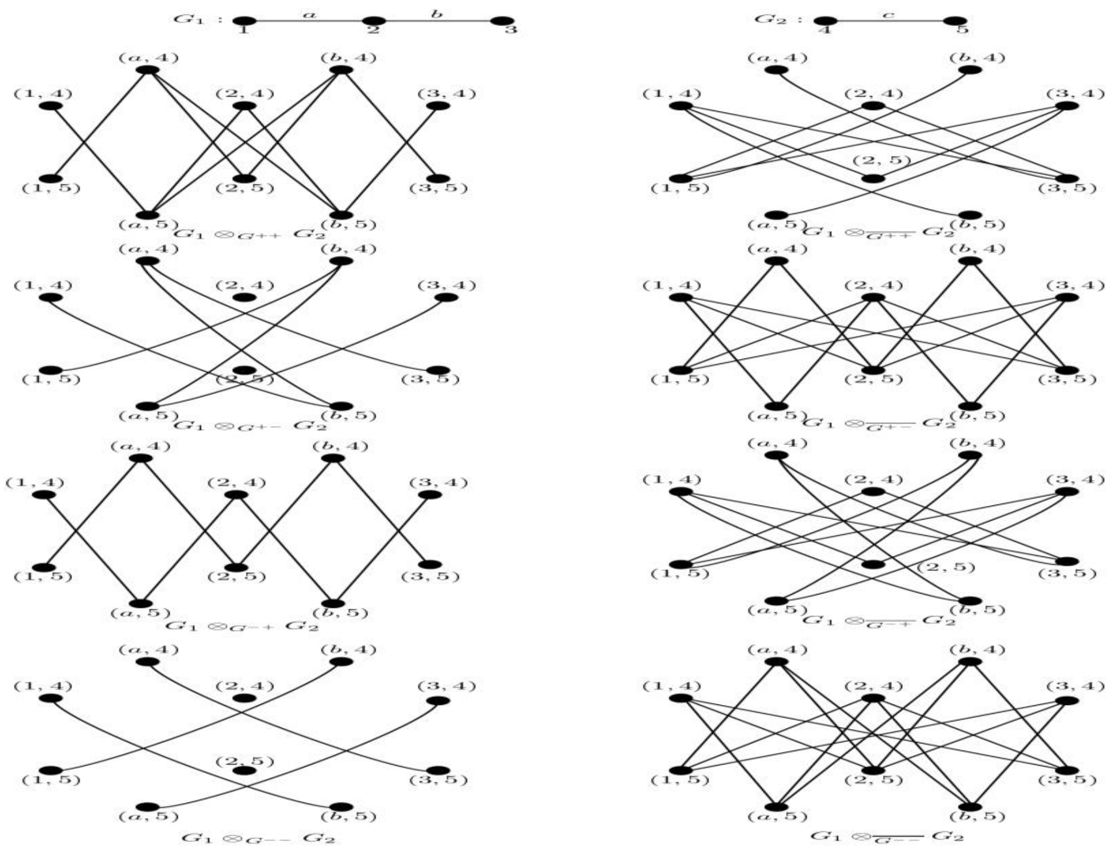

Figure 2: Graphs G_1, G_2 and $G_1 \otimes_{\mathcal{F}} G_2$.

The graph $G_1 \otimes_{G^{++}} G_2 \cong G_1 \otimes_{T_1} G_2$ and its first Zagreb index and second Zagreb index can be found in [8]. In this paper, we study the first Zagreb index and second Zagreb index of $G_1 \otimes_{G^{+-}} G_2$, $G_1 \otimes_{G^{-+}} G_2$, $G_1 \otimes_{G^{--}} G_2$, $G_1 \otimes_{\overline{G^{++}}} G_2$, $G_1 \otimes_{\overline{G^{+-}}} G_2$, $G_1 \otimes_{\overline{G^{-+}}} G_2$ and $G_1 \otimes_{\overline{G^{--}}} G_2$.

[2] PRELIMINARIES

The following results are useful for proving our main results.

Theorem 2.1 *[1, 10] Let G be a (n, m) graph. Then*

1. $M_1(\overline{G}) = M_1(G) + n(n-1)^2 - 4m(n-1)$,
2. $\overline{M_1}(G) = 2m(n-1) - M_1(G)$,
3. $\overline{M_1}(\overline{G}) = 2m(n-1) - M_1(G)$.

Lemma 2.2 *[8] Let G_1 and G_2 be two graphs. Then $M_2(G_1 \otimes_{\mathcal{F}} G_2) = 2M_2(\mathcal{F}(G_1))M_2(G_2)$.*
Theorem 2.3 *[4] Let G be a graph of order n and size m. Then*

- $M_2(T^{0+-}) = m^2((n-2)(n-6)+4) - m(n-2)^2 + \frac{M_1(G)}{2}(2(m+1)(n-4) + (n-2)^2) + F(G) + 2M_2(G) + (n-2)EM_1(G) + EM_2(G)$,
- $M_2(T^{0-+}) = (m+1)^2\binom{m+1}{2} - (\frac{m^2-5}{2})M_1(G) - (m+1)\overline{EM_1(G)} + \overline{EM_2(G)} - F(G) - 2M_2(G)$,

- $M_2(T^{0--}) = (n+m-3)^2(\binom{m+1}{2} - \frac{M_1(G)}{2}) + m^2(n-4)(n+m-1) - (m(n-5) - n+1)M_1(G) - F(G) - 2M_2(G) - (n+m-3)\overline{EM_1(G)} + \overline{EM_2(G)}$,
- $M_2(T^{1++}) = (n-1)^2(\binom{n}{2} + 2m) + 2m(m-2) + \frac{M_1(G)}{2}(4n-1) + F(G) + 2M_2(G) + 2EM_1(G) + EM_2(G)$,
- $M_2(T^{1+-}) = (n+m-1)^2(\frac{n}{2}) + m(n+m-1)((n-2)(n-4) - 2(n-1)) + 2M_2(G) + F(G) - 2m^2(n-5) - m(n-2)^2 + \frac{M_1(G)}{2}((n-2)^2 - 1) + M_1(G)((n-2)(n+m) - 2(m+1)) + (n-2)EM_1(G) + EM_2(G)$,
- $M_2(T^{1-+}) = (n-1)^2\binom{n}{2} + (m+1)^2\binom{m+1}{2} + 2m((n-1)(n+m+2) + m) - \frac{M_1(G)}{2}(m^2 + (4n-2)) - F(G) - 2M_2(G) - (m+1)\overline{EM_1(G)} + \overline{EM_2(G)}$,

$M_2(T^{1--}) = (n+m-1)^2(\frac{n}{2}) + (n+m-3)^2\binom{m+1}{2} - 2m^2(n+m-2) - 2M_2(G) - F(G) + m(n+m-1)((n-2)(n+m-1) - 2(n-1)) - \frac{M_1(G)}{2}((n+m-3)^2 + 1) - M_1(G)((n-2)(n+m-1) - (n+3m-1)) - (n+m-3)\overline{EM_1(G)} + \overline{EM_2(G)}$

Theorem 2.4 *[21] Let G be a graph of order n and size m. Then*

1. $M_2(\overline{G}) = \frac{1}{2}n(n-1)^3 - 3m(n-1)^2 + 2m^2 + \frac{2n-3}{2}M_1(G) - M_2(G)$,
2. $\overline{M_2}(G) = 2m^2 - \frac{1}{2}M_1(G) - M_2(G)$,
3. $\overline{M_2}(\overline{G}) = m(n-1)^2 - (n-1)M_1(G) + M_2(G)$.

Theorem 2.5 *[16] Let G be a graph with n vertices and m edges. Then*

1. $F(\overline{G}) = n(n-1)^3 - 4m(n-1)^2 + 3(n-1)M_1(G) - F(G)$
2. $\overline{F}(G) = (n-1)M_1(G) - F(G)$
3. $\overline{F}(\overline{G}) = 2m(n-1)^2 - 2(n-1)M_1(G) + F(G)$.

Proposition 2.6 *[7] Let G be a graph of order n, size m and let v be the point-vertex of $T^{xyz}(G)$ corresponding to a vertex v of G. Then*

$$d_{T^{xy+}}(v) = \begin{cases} d_G(v), & if\ \ x = 0, y \in \{0,1,+,-\}, \\ n-1+d_G(v), & if\ \ x = 1, y \in \{0,1,+,-\}, \\ 2d_G(v), & if\ \ x = +, y \in \{0,1,+,-\}, \\ n-1, & if\ \ x = -, y \in \{0,1,+,-\}, \end{cases}$$

$$d_{T^{xy-}}(v) = \begin{cases} m-d_G(v), & if\ \ x = 0, y \in \{0,1,+,-\}, \\ n+m-1-d_G(v), & if\ \ x = 1, y \in \{0,1,+,-\}, \\ m, & if\ \ x = +, y \in \{0,1,+,-\}, \\ n+m-1-2d_G(v), & if\ \ x = -, y \in \{0,1,+,-\}. \end{cases}$$

Proposition 2.7 *[7] Let G be a graph of order n, size m and let e be the line-vertex of $T^{xyz}(G)$ corresponding to a edge e of G. Then*

$$d_{T^{xy+}}(e) = \begin{cases} 2, & if \;\; y = 0, x \in \{0,1,+,-\}, \\ m+1, & if \;\; y = 1, x \in \{0,1,+,-\}, \\ 2 + d_G(e), & if \;\; y = +, x \in \{0,1,+,-\}, \\ m+1-d_G(e), & if \;\; y = -, x \in \{0,1,+,-\}, \end{cases}$$

$$d_{T^{xy-}}(e) = \begin{cases} n-2, & if \;\; y = 0, x \in \{0,1,+,-\}, \\ n+m-3, & if \;\; y = 1, x \in \{0,1,+,-\}, \\ n-2+d_G(e), & if \;\; y = +, x \in \{0,1,+,-\}, \\ n+m-3-d_G(e), & if \;\; y = -, x \in \{0,1,+,-\}. \end{cases}$$

Proposition 2.8 *[6] Let G be a graph of order n and size m. Then*

- $|V(T^{xyz}(G))| = n + m.$
- $|E(T^{xyz}(G))| = |E(G^x)| + |E(L(G)^y)| + |E(W)|$, where

$$|E(G^x)| = \begin{cases} 0, & if \;\; x = 0, \\ \binom{n}{2}, & if \;\; x = 1, \\ m, & if \;\; x = +, \\ \binom{n}{2} - m, & if \;\; x = -, \end{cases}$$

$$|E(L(G)^y)| = \begin{cases} 0, & if \;\; y = 0, \\ \binom{m}{2}, & if \;\; y = 1, \\ -m + \frac{1}{2}\sum_{i=1}^{n} d_G(v_i)^2, & if \;\; y = +, \\ \binom{m+1}{2} - \frac{1}{2}\sum_{i=1}^{n} d_G(v_i)^2, & if \;\; y = -, \end{cases}$$

$$|E(W)| = \begin{cases} 0, & if \;\; z = 0, \\ mn, & if \;\; z = 1, \\ 2m, & if \;\; z = +, \\ m(n-2), & if \;\; z = -. \end{cases}$$

T^{0++}, T^{0+-}, T^{0-+}, T^{0--}, T^{1--}, T^{1-+}, T^{1+-} and T^{1++} are nothing but G^{++}, G^{+-}, G^{-+}, G^{--}, $\overline{G^{++}}$, $\overline{G^{+-}}$, $\overline{G^{-+}}$ and $\overline{G^{--}}$ respectively.

[3] ZAGREB INDICES OF $\mathcal{F}$-TENSOR PRODUCTS OF GENERALIZED MIDDLE GRAPHS

We start by stating the following proposition which will be needed to prove our main results:

Proposition 3.1 *If G_1 and G_2 are (n_1, m_1) and (n_2, m_2) graphs respectively. Then $|V(G_1 \otimes_{\mathcal{F}} G_2)| = n_2(n_1 + m_1)$ and*

1. $|E(G_1 \otimes_{\overline{G^{++}}} G_2)| = [n_1(n_1 - 1) + m_1(m_1 + 2n_1 - 3) - M_1(G_1)]m_2,$
2. $|E(G_1 \otimes_{G^{+-}} G_2)| = [2m_1(n_1 - 3) + M_1(G_1)]m_2,$

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3. $|E(G_1 \otimes_{\overline{G^{+-}}} G_2)| = [n_1(n_1-1) + m_1(m_1+5) - M_1(G_1)]m_2$,
4. $|E(G_1 \otimes_{G^{-+}} G_2)| = [m_1(m_1+5) - M_1(G_1)]m_2$,
5. $|E(G_1 \otimes_{\overline{G^{-+}}} G_2)| = [n_1(n_1-1) + 2m_1(n_1-3) + M_1(G_1)]m_2$,
6. $|E(G_1 \otimes_{G^{--}} G_2)| = [m_1(m_1+2n_1-3) - M_1(G_1)]m_2$,
7. $|E(G_1 \otimes_{\overline{G^{--}}} G_2)| = [n_1(n_1-1) + 2m_1 + M_1(G_1)]m_2$.

Theorem 3.2 If G_1 and G_2 are (n_1, m_1) and (n_2, m_2) graphs respectively, then

$$M_1(G_1 \otimes_{\overline{G^{++}}} G_2) = M_1(G_2)(M_1(G_1) + EM_1(G_1) + (n_1+m_1-1)^2 n_1 + (n_1+m_1-3)^2 m_1 - 4(m_L(n_1+m_1-3) + m_1(n_1+m_1-1))).$$

Proof. Using the definition of first Zagreb index, Propositions 2.6 and 2.7, we have

$$\begin{aligned} M_1(G_1 \otimes_{\overline{G^{++}}} G_2) &= \sum_{(u,v)\in V(G_1 \otimes_{\overline{G^{++}}} G_2)} d^2_{G_1 \otimes_{\overline{G^{++}}} G_2}(u,v) \\ &= \sum_{u\in V(\overline{G^{++}}(G_1))\cap V(G_1)} \sum_{v\in V(G_2)} (d_{\overline{G^{++}}(G_1)}(u) d_{G_2}(v))^2 \\ &+ \sum_{z\in V(G_2)} \sum_{e\in V(\overline{G^{++}}(G_1))\cap E(G_1)} (d_{\overline{G^{++}}(G_1)}(e) d_{G_2}(z))^2. \end{aligned}$$

For $u \in V(\overline{G^{++}}(G_1)) \cap V(G_1)$, $d_{\overline{G^{++}}(G_1)}(u) = n_1 + m_1 - 1 - d_{G_1}(u)$ and

for $e \in V(\overline{G^{++}}(G_1)) \cap E(G_1)$, $d_{\overline{G^{++}}(G_1)}(e) = n_1 + m_1 - 3 - d_{G_1}(e)$.

Therefore,

$$\begin{aligned} M_1(G_1 \otimes_{\overline{G^{++}}} G_2) &= \sum_{u\in V(G_1)} \sum_{v\in V(G_2)} ((n_1+m_1-1-d_{G_1}(u)) d_{G_2}(v))^2 \\ &+ \sum_{z\in V(G_2)} \sum_{e\in E(G_1)} ((n_1+m_1-3-d_{G_1}(e)) d_{G_2}(z))^2 \\ &= ((n_1+m_1-1)^2 n_1 + M_1(G_1) - 4m_1(n_1+m_1-1)) M_1(G_2) \\ &+ ((n_1+m_1-3)^2 m_1 + EM_1(G_1) - 4m_L(n_1+m_1-3)) M_1(G_2) \\ &= M_1(G_2)(M_1(G_1) + EM_1(G_1) + (n_1+m_1-1)^2 n_1 \\ &+ (n_1+m_1-3)^2 m_1 - 4(m_L(n_1+m_1-3) + m_1(n_1+m_1-1))). \end{aligned}$$

Corollary 3.3 If G_1 and G_2 are (n_1, m_1) and (n_2, m_2) graphs respectively, then

$$\begin{aligned} M_1(\overline{G_1 \otimes_{\overline{G^{++}}} G_2}) &= M_1(G_2)(M_1(G_1) + EM_1(G_1) + (n_1+m_1-1)^2 n_1 \\ &+ (n_1+m_1-3)^2 m_1 - 4(m_L(n_1+m_1-3) + m_1(n_1+m_1-1))) \\ &+ (n_2(n_1+m_1) - 1)(n_2(n_1+m_1)(n_2(n_1+m_1) - 1) \\ &- 4(n_1(n_1-1) + m_1(2n_1+m_1-3) - M_1(G_1))). \end{aligned}$$

Proof. Using Theorem 3.2 and Proposition 3.1(i) in Theorem 2.1(i), we get the required result.

Corollary 3.4 If G_1 and G_2 are (n_1, m_1) and (n_2, m_2) graphs respectively, then

$$\begin{aligned} \overline{M_1}(G_1 \otimes_{\overline{G^{++}}} G_2) &= 2m_2(n_1(n_1-1) + m_1(2n_1+m_1-3) - M_1(G_1))(n_2(n_1+m_1) - 1) \\ &- M_1(G_2)(M_1(G_1) + EM_1(G_1) + (n_1+m_1-1)^2 n_1 \\ &+ (n_1+m_1-3)^2 m_1 - 4m_2(m_L(n_1+m_1-3) + m_1(n_1+m_1-1))). \end{aligned}$$

Proof. Using Theorem 3.2 and Proposition 3.1(i) in Theorem 2.1(ii), we get the required result.

Corollary 3.5 If G_1 and G_2 are (n_1, m_1) and (n_2, m_2) graphs respectively, then

$$\begin{aligned} \overline{M_1}(\overline{G_1 \otimes_{\overline{G^{++}}} G_2}) &= 2m_2(n_1(n_1-1) + m_1(2n_1+m_1-3) - M_1(G_1))(n_2(n_1+m_1) - 1) \\ &- M_1(G_2)(M_1(G_1) + EM_1(G_1) + (n_1+m_1-1)^2 n_1 \\ &+ (n_1+m_1-3)^2 m_1 - 4(m_L(n_1+m_1-3) + m_1(n_1+m_1-1))). \end{aligned}$$

Proof. Using Theorem 3.2 and Proposition 3.1(i) in Theorem 2.1(iii), we get the required result.

Theorem 3.6 If G_1 and G_2 are (n_1, m_1) and (n_2, m_2) graphs respectively, then

$$M_1(G_1 \otimes_{G^{+-}} G_2) = M_1(G_2)(M_1(G_1) + EM_1(G_1) + m_1^2(n_1 - 4) \\ +(n_1 - 2)(m_1(n_1 - 2) + 4m_L)).$$

Proof. Using the definition of first Zagreb index, Propositions 2.6 and 2.7, we have

$$M_1(G_1 \otimes_{G^{+-}} G_2) = \sum_{(u,v)\in V(G_1 \otimes_{G^{+-}} G_2)} d^2_{G_1 \otimes_{G^{+-}} G_2}(u, v) \\ = \sum_{u\in V(G^{+-}(G_1))\cap V(G_1)} \sum_{v\in V(G_2)} (d_{G^{+-}(G_1)}(u) d_{G_2}(v))^2 \\ + \sum_{z\in V(G_2)} \sum_{e\in V(G^{+-}(G_1))\cap E(G_1)} (d_{G^{+-}(G_1)}(e) d_{G_2}(z))^2.$$

For $u \in V(G^{+-}(G_1)) \cap V(G_1)$, $d_{G^{+-}(G_1)}(u) = m_1 - d_{G_1}(u)$ and
for $e \in V(G^{+-}(G_1)) \cap E(G_1)$, $d_{G^{+-}(G_1)}(e) = n_1 - 2 + d_{G_1}(e)$.
Therefore,

$$M_1(G_1 \otimes_{G^{+-}} G_2) = \sum_{u\in V(G_1)} \sum_{v\in V(G_2)} ((m_1 - d_{G_1}(u)) d_{G_2}(v))^2 \\ + \sum_{z\in V(G_2)} \sum_{e\in E(G_1)} ((n_1 - 2 + d_{G_1}(e)) d_{G_2}(z))^2 \\ = (m_1^2 n_1 + M_1(G_1) - 4m_1^2) M_1(G_2) + ((n_1 - 2)^2 m_1 \\ + EM_1(G_1) + 4m_L(n_1 - 2)) M_1(G_2) \\ = M_1(G_2)(M_1(G_1) + EM_1(G_1) + m_1^2(n_1 - 4) \\ +(n_1 - 2)(m_1(n_1 - 2) + 4m_L)).$$

Corollary 3.7 If G_1 and G_2 are (n_1, m_1) and (n_2, m_2) graphs respectively, then

$$M_1(\overline{G_1 \otimes_{G^{+-}} G_2}) = M_1(G_2)(M_1(G_1) + EM_1(G_1) + m_1^2(n_1 - 4) \\ +(n_1 - 2)(m_1(n_1 - 2) + 4m_L)) \\ +(n_2(n_1 + m_1) - 1)(n_2(n_1 + m_1)(n_2(n_1 + m_1) - 1) \\ -4m_2(2m_1(n_1 - 3) + M_1(G_1))).$$

Proof. Using Theorem 3.6 and Proposition 3.1(ii) in Theorem 2.1(i), we get the required result.

Corollary 3.8 If G_1 and G_2 are (n_1, m_1) and (n_2, m_2) graphs respectively, then

$$\overline{M_1}(G_1 \otimes_{G^{+-}} G_2) = 2m_2(2m_1(n_1 - 3) + M_1(G_1))(n_2(n_1 + m_1) - 1) \\ -M_1(G_2)(M_1(G_1) + EM_1(G_1) + m_1^2(n_1 - 4) \\ +(n_1 - 2)(m_1(n_1 - 2) + 4m_L)).$$

Proof. Using Theorem 3.6 and Proposition 3.1(ii) in Theorem 2.1(ii), we get the required result.

Corollary 3.9 If G_1 and G_2 are (n_1, m_1) and (n_2, m_2) graphs respectively, then

$$\overline{M_1}(\overline{G_1 \otimes_{G^{+-}} G_2}) = 2m_2(2m_1(n_1 - 3) + M_1(G_1))(n_2(n_1 + m_1) - 1) \\ -M_1(G_2)(M_1(G_1) + EM_1(G_1) + m_1^2(n_1 - 4) \\ +(n_1 - 2)(m_1(n_1 - 2) + 4m_L)).$$

Proof. Using Theorem 3.6 and Proposition 3.1(ii) in Theorem 2.1(iii), we get the required result.

Theorem 3.10 If G_1 and G_2 are (n_1, m_1) and (n_2, m_2) graphs respectively, then

$$M_1(G_1 \otimes_{\overline{G^{+-}}} G_2) = M_1(G_2)(M_1(G_1) + EM_1(G_1) + (n_1 - 1)^2 n_1 \\ +(m_1 + 1)^2 m_1 + 4(m_1(n_1 - 1) - m_L(m_1 + 1))).$$

Proof. Using the definition of first Zagreb index, Propositions 2.6 and 2.7, we have

$$M_1(G_1 \otimes_{\overline{G^{+-}}} G_2) = \sum_{(u,v)\in V(G_1 \otimes_{\overline{G^{+-}}} G_2)} d^2_{G_1 \otimes_T G_2}(u,v)$$
$$= \sum_{u\in V(T(G_1))\cap V(G_1)} \sum_{v\in V(G_2)} (d_{\overline{G^{+-}}(G_1)}(u) d_{G_2}(v))^2$$
$$+ \sum_{z\in V(G_2)} \sum_{e\in V(\overline{G^{+-}}(G_1))\cap E(G_1)} (d_{\overline{G^{+-}}(G_1)}(e) d_{G_2}(z))^2.$$

For $u \in V(\overline{G^{+-}}(G_1)) \cap V(G_1)$, $d_{\overline{G^{+-}}(G_1)}(u) = n_1 - 1 + d_{G_1}(u)$ and
for $e \in V(\overline{G^{+-}}(G_1)) \cap E(G_1)$, $d_{\overline{G^{+-}}(G_1)}(e) = m_1 + 1 - d_{G_1}(e)$.
Therefore,

$$M_1(G_1 \otimes_{\overline{G^{+-}}} G_2) = \sum_{u\in V(G_1)} \sum_{v\in V(G_2)} ((n_1 - 1 + d_{G_1}(u)) d_{G_2}(v))^2$$
$$+ \sum_{z\in V(G_2)} \sum_{xy\in E(G_1)} ((m_1 + 1 - d_{G_1}(e)) d_{G_2}(z))^2$$
$$= ((n_1-1)^2 n_1 + M_1(G_1) + 4m_1(n_1-1))M_1(G_2)$$
$$+((m_1+1)^2 m_1 + EM_1(G_1) - 4m_L(m_1+1))M_1(G_2)$$
$$= M_1(G_2)(M_1(G_1) + EM_1(G_1) + (n_1-1)^2 n_1$$
$$+(m_1+1)^2 m_1 + 4(m_1(n_1-1) - m_L(m_1+1))).$$

Corollary 3.11 If G_1 and G_2 are (n_1, m_1) and (n_2, m_2) graphs respectively, then

$$\overline{M_1(G_1 \otimes_{\overline{G^{+-}}} G_2)} = M_1(G_2)(M_1(G_1) + EM_1(G_1) + (n_1-1)^2 n_1$$
$$+(m_1+1)^2 m_1 + 4(m_1(n_1-1) - m_L(m_1+1)))$$
$$+(n_2(n_1+m_1)-1)(n_2(n_1+m_1)(n_2(n_1+m_1)-1)$$
$$-4m_2(n_1(n_1-1) + m_1(m_1+5) - M_1(G_1))).$$

Proof. Using Theorem 3.14 and Proposition 3.1(iii) in Theorem 2.1(i), we get the required result.

Corollary 3.12 If G_1 and G_2 are (n_1, m_1) and (n_2, m_2) graphs respectively, then

$$\overline{M_1}(G_1 \otimes_{\overline{G^{+-}}} G_2) = 2m_2(n_1(n_1-1) + m_1(m_1+5) - M_1(G_1))(n_2(n_1+m_1)-1)$$
$$-M_1(G_2)(M_1(G_1) + EM_1(G_1) + (n_1-1)^2 n_1$$
$$+(m_1+1)^2 m_1 + 4(m_1(n_1-1) - m_L(m_1+1))).$$

Proof. Using Theorem 3.14 and Proposition 3.1(iii) in Theorem 2.1(ii), we get the required result.

Corollary 3.13 If G_1 and G_2 are (n_1, m_1) and (n_2, m_2) graphs respectively, then

$$\overline{M_1(G_1 \otimes_{\overline{G^{+-}}} G_2)} = 2m_2(n_1(n_1-1) + m_1(m_1+5) - M_1(G_1))(n_2(n_1+m_1)-1)$$
$$-M_1(G_2)(M_1(G_1) + EM_1(G_1) + (n_1-1)^2 n_1$$
$$+(m_1+1)^2 m_1 + 4(m_1(n_1-1) - m_L(m_1+1))).$$

Proof. Using Theorem 3.14 and Proposition 3.1(iii) in Theorem 2.1(iii), we get the required result.

Theorem 3.14 If G_1 and G_2 are (n_1, m_1) and (n_2, m_2) graphs respectively, then

$$M_1(G_1 \otimes_{G^{-+}} G_2) = M_1(G_2)(M_1(G_1) + EM_1(G_1) + (m_1+1)^2 m_1 - 4m_L(m_1+1)).$$

Proof. Using the definition of first Zagreb index, Propositions 2.6 and 2.7, we have

$$M_1(G_1 \otimes_{G^{-+}} G_2) = \sum_{(u,v)\in V(G_1 \otimes_{G^{-+}} G_2)} d^2_{G_1 \otimes_{G^{-+}} G_2}(u,v)$$
$$= \sum_{u\in V(G^{-+}(G_1))\cap V(G_1)} \sum_{v\in V(G_2)} (d_{G^{-+}(G_1)}(u) d_{G_2}(v))^2$$
$$+ \sum_{z\in V(G_2)} \sum_{e\in V(G^{-+}(G_1))\cap E(G_1)} (d_{G^{-+}(G_1)}(e) d_{G_2}(z))^2.$$

For $u \in V(G^{-+}(G_1)) \cap V(G_1)$, $d_{G^{-+}(G_1)}(u) = d_{G_1}(u)$ and
for $e \in V(G^{-+}(G_1)) \cap E(G_1)$, $d_{G^{-+}(G_1)}(e) = m_1 + 1 - d_{G_1}(e)$.
Therefore,

$$M_1(G_1 \otimes_{G^{-+}} G_2) = \sum_{u\in V(G_1)} \sum_{v\in V(G_2)} (d_{G_1}(u) d_{G_2}(v))^2$$
$$+ \sum_{z\in V(G_2)} \sum_{xy\in E(G_1)} ((m_1 + 1 - d_{G_1}(e)) d_{G_2}(z))^2$$

$$= M_1(G_1)M_1(G_2) + ((m_1+1)^2 m_1 + EM_1(G_1) - 4m_L(m_1+1))M_1(G_2)$$
$$= M_1(G_2)(M_1(G_1) + EM_1(G_1) + (m_1+1)^2 m_1 - 4m_L(m_1+1)).$$

Corollary 3.15 If G_1 and G_2 are (n_1, m_1) and (n_2, m_2) graphs respectively, then

$$\overline{M_1(G_1 \otimes_{G^{-+}} G_2)} = M_1(G_2)(M_1(G_1) + EM_1(G_1) + (m_1+1)^2 m_1 - 4m_L(m_1+1)) + (n_2(n_1+m_1)-1)(n_2(n_1+m_1)(n_2(n_1+m_1)-1) - 4m_2(m_1(m_1+5) - M_1(G_1))).$$

Proof. Using Theorem 3.14 and Proposition 3.1(iv) in Theorem 2.1(i), we get the required result.

Corollary 3.16 If G_1 and G_2 are (n_1, m_1) and (n_2, m_2) graphs respectively, then

$$\overline{M_1}(G_1 \otimes_{G^{-+}} G_2) = 2m_2(m_1(m_1+5) - M_1(G_1))(n_2(n_1+m_1)-1) - M_1(G_2)(M_1(G_1) + EM_1(G_1) + (m_1+1)^2 m_1 - 4m_L(m_1+1)).$$

Proof. Using Theorem 3.14 and Proposition 3.1(iv) in Theorem 2.1(ii), we get the required result.

Corollary 3.17 If G_1 and G_2 are (n_1, m_1) and (n_2, m_2) graphs respectively, then

$$\overline{M_1}\overline{(G_1 \otimes_{G^{-+}} G_2)} = 2m_2(m_1(m_1+5) - M_1(G_1))(n_2(n_1+m_1)-1) - M_1(G_2)(M_1(G_1) + EM_1(G_1) + (m_1+1)^2 m_1 - 4m_L(m_1+1)).$$

Proof. Using Theorem 3.14 and Proposition 3.1(iv) in Theorem 2.1(iii), we get the required result.

Theorem 3.18 If G_1 and G_2 are (n_1, m_1) and (n_2, m_2) graphs respectively, then

$$M_1(G_1 \otimes_{\overline{G^{-+}}} G_2) = M_1(G_2)(M_1(G_1) + EM_1(G_1) + (n_1+m_1-1)(n_1(n_1+m_1-1) - 4m_1) + (n_1-2)(m_1(n_1-2) + 4m_L)).$$

Proof. Using the definition of first Zagreb index, Propositions 2.6 and 2.7, we have

$$M_1(G_1 \otimes_{\overline{G^{-+}}} G_2) = \sum_{(u,v)\in V(G_1 \otimes_{\overline{G^{-+}}} G_2)} d^2_{G_1 \otimes_{\overline{G^{-+}}} G_2}(u,v)$$
$$= \sum_{u\in V(\overline{G^{-+}}(G_1))\cap V(G_1)} \sum_{v\in V(G_2)} (d_{\overline{G^{-+}}(G_1)}(u) d_{G_2}(v))^2 + \sum_{z\in V(G_2)} \sum_{e\in V(\overline{G^{-+}}(G_1))\cap E(G_1)} (d_{\overline{G^{-+}}(G_1)}(e) d_{G_2}(z))^2.$$

For $u \in V(\overline{G^{-+}}(G_1)) \cap V(G_1)$, $d_{\overline{G^{-+}}(G_1)}(u) = n_1 + m_1 - 1 - d_{G_1}(u)$ and
for $e \in V(\overline{G^{-+}}(G_1)) \cap E(G_1)$, $d_{\overline{G^{-+}}(G_1)}(e) = n_1 - 2 + d_{G_1}(e)$.
Therefore,

$$M_1(G_1 \otimes_{\overline{G^{-+}}} G_2) = \sum_{u\in V(G_1)} \sum_{v\in V(G_2)} ((n_1+m_1-1-d_{G_1}(u))d_{G_2}(v))^2 + \sum_{z\in V(G_2)} \sum_{e\in E(G_1)} ((n_1-2+d_{G_1}(e))d_{G_2}(z))^2$$
$$= ((n_1+m_1-1)^2 n_1 + M_1(G_1) - 4(n_1+m_1-1)m_1)M_1(G_2) + ((n_1-2)^2 m_1 + EM_1(G_1) + 4m_L(n_1-2))M_1(G_2)$$
$$= M_1(G_2)(M_1(G_1) + EM_1(G_1) + (n_1+m_1-1)(n_1(n_1+m_1-1) - 4m_1) + (n_1-2)(m_1(n_1-2) + 4m_L)).$$

Corollary 3.19 If G_1 and G_2 are (n_1, m_1) and (n_2, m_2) graphs respectively, then

$$M_1(\overline{G_1 \otimes_{\overline{G^{-+}}} G_2}) = M_1(G_2)(M_1(G_1) + EM_1(G_1) + (n_1+m_1-1)(n_1(n_1+m_1-1) - 4m_1) + (n_1-2)(m_1(n_1-2)+4m_L)) + (n_2(n_1+m_1)-1)(n_2(n_1+m_1)(n_2(n_1+m_1)-1) - 4m_2(n_1(n_1-1) + M_1(G_1) + 2m_1(n_1-3))).$$

Proof. Using Theorem 3.18 and Proposition 3.1(v) in Theorem 2.1(i), we get the required result.

Corollary 3.20 If G_1 and G_2 are (n_1, m_1) and (n_2, m_2) graphs respectively, then

$$\overline{M_1}(G_1 \otimes_{\overline{G^{-+}}} G_2) = 2m_2(n_1(n_1-1) + M_1(G_1) + 2m_1(n_1-3))(n_2(n_1+m_1)-1) - M_1(G_2)(M_1(G_1) + EM_1(G_1) + (n_1+m_1-1)(n_1(n_1+m_1-1)-4m_1) + (n_1-2)(m_1(n_1-2)+4m_L)).$$

Proof. Using Theorem 3.18 and Proposition 3.1(v) in Theorem 2.1(ii), we get the required result.

Corollary 3.21 If G_1 and G_2 are (n_1, m_1) and (n_2, m_2) graphs respectively, then

$$\overline{M_1}(\overline{G_1 \otimes_{\overline{G^{-+}}} G_2}) = 2m_2(n_1(n_1-1) + M_1(G_1) + 2m_1(n_1-3))(n_2(n_1+m_1)-1) - M_1(G_2)(M_1(G_1) + EM_1(G_1) + (n_1+m_1-1)(n_1(n_1+m_1-1)-4m_1) + (n_1-2)(m_1(n_1-2)+4m_L)).$$

Proof. Using Theorem 3.18 and Proposition 3.1(v) in Theorem 2.1(iii), we get the required result.

Theorem 3.22 If G_1 and G_2 are (n_1, m_1) and (n_2, m_2) graphs respectively, then

$$M_1(G_1 \otimes_{G^{--}} G_2) = M_1(G_2)(M_1(G_1) + EM_1(G_1) + m_1(n_1m_1 + (n_1+m_1-3)^2) - 4(m_1^2 + m_L(n_1+m_1-3))).$$

Proof. Using the definition of first Zagreb index, Propositions 2.6 and 2.7, we have

$$\begin{aligned} M_1(G_1 \otimes_{G^{--}} G_2) &= \sum_{(u,v)\in V(G_1\otimes_{G^{--}} G_2)} d^2_{G_1\otimes_{G^{--}} G_2}(u,v) \\ &= \sum_{u\in V(G^{--}(G_1))\cap V(G_1)} \sum_{v\in V(G_2)} (d_{G^{--}(G_1)}(u) d_{G_2}(v))^2 \\ &+ \sum_{z\in V(G_2)} \sum_{e\in V(G^{--}(G_1))\cap E(G_1)} (d_{G^{--}(G_1)}(e) d_{G_2}(z))^2. \end{aligned}$$

For $u \in V(G^{--}(G_1)) \cap V(G_1)$, $d_{G^{--}(G_1)}(u) = m_1 - d_{G_1}(u)$ and
for $e \in V(G^{--}(G_1)) \cap E(G_1)$, $d_{G^{--}(G_1)}(e) = n_1 + m_1 - 3 - d_{G_1}(e)$.
Therefore,

$$\begin{aligned} M_1(G_1 \otimes_{G^{--}} G_2) &= \sum_{u\in V(G_1)} \sum_{v\in V(G_2)} ((m_1 - d_{G_1}(u)) d_{G_2}(v))^2 \\ &+ \sum_{z\in V(G_2)} \sum_{e\in E(G_1)} ((n_1+m_1-3-d_{G_1}(e)) d_{G_2}(z))^2 \\ &= (m_1^2 n_1 + M_1(G_1) - 4m_1^2) M_1(G_2) + ((n_1+m_1-3)^2 m_1 \\ &+ EM_1(G_1) - 4m_L(n_1+m_1-3)) M_1(G_2) \\ &= M_1(G_2)(M_1(G_1) + EM_1(G_1) + m_1(n_1m_1 + (n_1+m_1-3)^2) \\ &- 4(m_1^2 + m_L(n_1+m_1-3))). \end{aligned}$$

Corollary 3.23 If G_1 and G_2 are (n_1, m_1) and (n_2, m_2) graphs respectively, then

$$\begin{aligned} M_1(\overline{G_1 \otimes_{G^{--}} G_2}) &= M_1(G_2)(M_1(G_1) + EM_1(G_1) + m_1(n_1m_1 + (n_1+m_1-3)^2) \\ &- 4(m_1^2 + m_L(n_1+m_1-3))) \\ &+ (n_2(n_1+m_1)-1)(n_2(n_1+m_1)(n_2(n_1+m_1)-1) \\ &- 4m_2(m_1(m_1+2n_1-3) - M_1(G_1))). \end{aligned}$$

Proof. Using Theorem 3.22 and Proposition 3.1(vi) in Theorem 2.1(i), we get the required result.

Corollary 3.24 If G_1 and G_2 are (n_1, m_1) and (n_2, m_2) graphs respectively, then

$$\begin{aligned}\overline{M_1}(G_1 \otimes_{G^{--}} G_2) &= 2m_2(m_1(m_1 + 2n_1 - 3) - M_1(G_1))(n_2(n_1 + m_1) - 1)\\ &-M_1(G_2)(M_1(G_1) + EM_1(G_1) + m_1(n_1m_1 + (n_1 + m_1 - 3)^2)\\ &-4(m_1^2 + m_L(n_1 + m_1 - 3))).\end{aligned}$$

Proof. Using Theorem 3.22 and Proposition 3.1(vi) in Theorem 2.1(ii), we get the required result.

Corollary 3.25 If G_1 and G_2 are (n_1, m_1) and (n_2, m_2) graphs respectively, then

$$\begin{aligned}\overline{M_1(G_1 \otimes_{G^{--}} G_2)} &= 2m_2(m_1(m_1 + 2n_1 - 3) - M_1(G_1))(n_2(n_1 + m_1) - 1)\\ &-M_1(G_2)(M_1(G_1) + EM_1(G_1) + m_1(n_1m_1 + (n_1 + m_1 - 3)^2)\\ &-4(m_1^2 + m_L(n_1 + m_1 - 3))).\end{aligned}$$

Proof. Using Theorem 3.22 and Proposition 3.1(vi) in Theorem 2.1(iii), we get the required result.

Theorem 3.26 If G_1 and G_2 are (n_1, m_1) and (n_2, m_2) graphs respectively, then

$$M_1(G_1 \otimes_{\overline{G^{--}}} G_2) = M_1(G_2)(M_1(G_1) + EM_1(G_1) + (n_1 - 1)^2 n_1 + 4(m_1 n_1 + m_L)).$$

Proof. Using the definition of first Zagreb index, Propositions 2.6 and 2.7, we have

$$\begin{aligned}M_1(G_1 \otimes_{\overline{G^{--}}} G_2) &= \textstyle\sum_{(u,v)\in V(G_1 \otimes_{\overline{G^{--}}} G_2)} d^2_{G_1 \otimes_{\overline{G^{--}}} G_2}(u, v)\\ &= \textstyle\sum_{u\in V(\overline{G^{--}}(G_1))\cap V(G_1)} \sum_{v\in V(G_2)} (d_{\overline{G^{--}}(G_1)}(u) d_{G_2}(v))^2\\ &+ \textstyle\sum_{z\in V(G_2)} \sum_{e\in V(\overline{G^{--}}(G_1))\cap E(G_1)} (d_{\overline{G^{--}}(G_1)}(e) d_{G_2}(z))^2.\end{aligned}$$

For $u \in V(\overline{G^{--}}(G_1)) \cap V(G_1)$, $d_{\overline{G^{--}}(G_1)}(u) = n_1 - 1 + d_{G_1}(u)$ and
for $e \in V(\overline{G^{--}}(G_1)) \cap E(G_1)$, $d_{\overline{G^{--}}(G_1)}(e) = 2 + d_{G_1}(e)$.

Therefore,

$$\begin{aligned}M_1(G_1 \otimes_{\overline{G^{--}}} G_2) &= \textstyle\sum_{u\in V(G_1)} \sum_{v\in V(G_2)} ((n_1 - 1 + d_{G_1}(u)) d_{G_2}(v))^2\\ &+ \textstyle\sum_{z\in V(G_2)} \sum_{e\in E(G_1)} ((2 + d_{G_1}(e)) d_{G_2}(z))^2\\ &= ((n_1 - 1)^2 n_1 + M_1(G_1) + 4m_1(n_1 - 1))M_1(G_2)\\ &+(4m_1 + EM_1(G_1) + 8m_L)M_1(G_2)\\ &= M_1(G_2)(M_1(G_1) + EM_1(G_1) + (n_1 - 1)^2 n_1 + 4(m_1 n_1 + m_L)).\end{aligned}$$

Corollary 3.27 If G_1 and G_2 are (n_1, m_1) and (n_2, m_2) graphs respectively, then

$$\begin{aligned}M_1(\overline{G_1 \otimes_{\overline{G^{--}}} G_2}) &= M_1(G_2)(M_1(G_1) + EM_1(G_1) + (n_1 - 1)^2 n_1 + 4(m_1 n_1 + m_L))\\ &+(n_2(n_1 + m_1) - 1)(n_2(n_1 + m_1)(n_2(n_1 + m_1) - 1)\\ &-4m_2(n_1(n_1 - 1) + 2m_1 + M_1(G_1))).\end{aligned}$$

Proof. Using Theorem 3.26 and Proposition 3.1(vii) in Theorem 2.1(i), we get the required result.

Corollary 3.28 If G_1 and G_2 are (n_1, m_1) and (n_2, m_2) graphs respectively, then

$$\begin{aligned}\overline{M_1}(G_1 \otimes_{\overline{G^{--}}} G_2) &= 2m_2(n_1(n_1 - 1) + 2m_1 + M_1(G_1))(n_2(n_1 + m_1) - 1)\\ &-M_1(G_2)(M_1(G_1) + EM_1(G_1) + (n_1 - 1)^2 n_1 + 4(m_1 n_1 + m_L)).\end{aligned}$$

Proof. Using Theorem 3.26 and Proposition 3.1(vii) in Theorem 2.1(ii), we get the required result.

Corollary 3.29 If G_1 and G_2 are (n_1, m_1) and (n_2, m_2) graphs respectively, then

$$\overline{M_1}(\overline{G_1 \otimes_{\overline{G^{--}}} G_2}) = 2m_2(n_1(n_1-1)+2m_1+M_1(G_1))(n_2(n_1+m_1)-1) \\ -M_1(G_2)(M_1(G_1)+EM_1(G_1)+(n_1-1)^2n_1+4(m_1n_1+m_L)).$$

Proof. Using Theorem 3.26 and Proposition 3.1(vii) in Theorem 2.1(iii), we get the required result.

From Theorem 2.4, it is clear that if $M_1(G_1 \otimes_{\mathcal{F}} G_2)$ and $M_2(G_1 \otimes_{\mathcal{F}} G_2)$ are known, then $M_2(\overline{G_1 \otimes_{\mathcal{F}} G_2})$, $\overline{M_2}(G_1 \otimes_{\mathcal{F}} G_2)$ and $\overline{M_2}(\overline{G_1 \otimes_{\mathcal{F}} G_2})$ are known. As some expressions for $M_1(G_1 \otimes_{\mathcal{F}} G_2)$ are known by Theorems 3.2, 3.6, 3.10, 3.14, 3.18, 3.22 and 3.26 what really needs to be calculated are some expressions for $M_2(G_1 \otimes_{\mathcal{F}} G_2)$. This we do now:

Theorem 3.30 If G_1 and G_2 are (n_1, m_1) and (n_2, m_2) graphs respectively, then

$$M_2(G_1 \otimes_{\overline{G^{++}}} G_2) = 2((n_1+m_1-1)^2\binom{n_1}{2}+(n_1+m_1-3)^2\binom{m_1+1}{2}-2m_1^2(n_1+m_1-2)-2M_2(G_1)-F(G_1)+m_1(n_1+m_1-1)((n_1-2)(n_1+m_1-1)-2(n_1-1))-\frac{M_1(G_1)}{2}((n_1+m_1-3)^2+1)-M_1(G_1)((n_1-2)(n_1+m_1-1)-(n_1+3m_1-1))-(n_1+m_1-3)\overline{EM_1(G_1)}+\overline{EM_2(G_1)})M_2(G_2)$$

$$M_2(G_1 \otimes_{G^{+-}} G_2) = 2(m_1^2((n_1-2)(n_1-6)+4)-m_1(n_1-2)^2+\frac{M_1(G_1)}{2}(2(m_1+1)(n_1-4)+(n_1-2)^2)+F(G_1)+2M_2(G_1)+(n_1-2)EM_1(G_1)+EM_2(G_1))M_2(G_2)$$

$$M_2(G_1 \otimes_{\overline{G^{+-}}} G_2) = 2((n_1-1)^2\binom{n_1}{2}+(m_1+1)^2\binom{m_1+1}{2}+2m_1((n_1-1)(n_1+m_1+2)+m_1)-\frac{M_1(G_1)}{2}(m_1^2+(4n_1-2))-F(G_1)-2M_2(G_1)-(m_1+1)\overline{EM_1(G_1)}+\overline{EM_2(G_1)})M_2(G_2)$$

$$M_2(G_1 \otimes_{G^{-+}} G_2) = 2((m_1+1)^2\binom{m_1+1}{2}-(\frac{m_1^2-5}{2})M_1(G_1)-(m_1+1)\overline{EM_1(G_1)}+\overline{EM_2(G_1)}-F(G_1)-2M_2(G_1))M_2(G_2)$$

$$M_2(G_1 \otimes_{\overline{G^{-+}}} G_2) = 2((n_1+m_1-1)^2\binom{n_1}{2}+m_1(n_1+m_1-1)((n_1-2)(n_1-4)-2(n_1-1))+2M_2(G_1)+F(G_1)-2m_1^2(n_1-5)-m_1(n_1-2)^2+\frac{M_1(G_1)}{2}((n_1-2)^2-1)+M_1(G_1)((n_1-2)(n_1+m_1)-2(m_1+1))+(n_1-2)EM_1(G_1)+EM_2(G_1))M_2(G_2)$$

$$M_2(G_1 \otimes_{G^{--}} G_2) = 2((n_1+m_1-3)^2(\binom{m_1+1}{2}-\frac{M_1(G)}{2})+m_1^2(n_1-4)(n_1+m_1-1)-(m_1(n_1-5)-n_1+1)M_1(G_1)-F(G_1)-2M_2(G_1)-(n_1+m_1-3)\overline{EM_1(G_1)}+\overline{EM_2(G_1)})M_2(G_2)$$

$$M_2(G_1 \otimes_{\overline{G^{--}}} G_2) = 2((n_1-1)^2(\binom{n_1}{2}+2m_1)+2m_1(m_1-2)+\frac{M_1(G)}{2}(4n_1-1)+F(G_1)+2M_2(G_1)+2EM_1(G_1)+EM_2(G_1))M_2(G_2)$$

Proof. Using Theorem 2.3 and Lemma 2.2, we get the desired results.

[4] F-INDEX AND COINDEX OF $\mathcal{F}$-TENSOR PRODUCTS OF GENERALIZED MIDDLE GRAPHS

In this section, we proceed to obtain F-index and coindex of $\mathcal{F}$-tensor products of generalized middle graphs and their complements.

Theorem 4.1 *If G_1 and G_2 are (n_1, m_1) and (n_2, m_2) graphs respectively, then*

$$F(G_1 \otimes_{\overline{G^{++}}} G_2) = F(G_2)((n_1+m_1-1)^3 n_1 - F(G_1) - 6(n_1+m_1-1)^2 m_1 +3(n_1+m_1-1)M_1(G_1) + (n_1+m_1-3)^3 m_1 - EF(G_1) -6m_L(n_1+m_1-3)^2 + 3(n_1+m_1-3)EM_1(G_1)).$$

Proof. Using the definition of F-index, Propositions 2.6 and 2.7, we obtain desired result.

Corollary 4.2 *If G_1 and G_2 are (n_1, m_1) and (n_2, m_2) graphs respectively, then*

$$F(\overline{G_1 \otimes_{\overline{G^{++}}} G_2}) = n_2(n_1+m_1)(n_2(n_1+m_1)-1)^3 - 4m_2(n_1(n_1-1) +m_1(m_1+2n_1-3) - M_1(G_1))(n_2(n_1+m_1)-1)^2 +3(n_2(n_1+m_1)-1)M_1(G_2)(M_1(G_1)+EM_1(G_1) +(n_1+m_1-1)^2 n_1 + (n_1+m_1-3)^2 m_1 -4(m_L(n_1+m_1-3)+m_1(n_1+m_1-1))) -F(G_2)((n_1+m_1-1)^3 n_1 - F(G_1) - 6(n_1+m_1-1)^2 m_1 +3(n_1+m_1-1)M_1(G_1) + (n_1+m_1-3)^3 m_1 - EF(G_1) -6m_L(n_1+m_1-3)^2 + 3(n_1+m_1-3)EM_1(G_1)).$$

Proof. Using Proposition 3.1, Theorems 3.2 and 2.5 (i) we get the desired result.

Corollary 4.3 *If G_1 and G_2 are (n_1, m_1) and (n_2, m_2) graphs respectively, then*

$$\overline{F}(G_1 \otimes_{\overline{G^{++}}} G_2) = (n_2(n_1+m_1)-1)M_1(G_2)(M_1(G_1)+EM_1(G_1)+4(m_1+2m_L)) -F(G_2)((n_1+m_1-1)^3 n_1 - F(G_1) - 6(n_1+m_1-1)^2 m_1 +3(n_1+m_1-1)M_1(G_1) + (n_1+m_1-3)^3 m_1 - EF(G_1) -6m_L(n_1+m_1-3)^2 + 3(n_1+m_1-3)EM_1(G_1)).$$

Proof. Using Proposition 3.1, Theorems 3.2 and 2.5 (ii) we get the desired result.

Corollary 4.4 *If G_1 and G_2 are (n_1, m_1) and (n_2, m_2) graphs respectively, then*

$$\overline{F}(\overline{G_1 \otimes_{\overline{G^{++}}} G_2}) = 2m_2(n_1(n_1-1)+m_1(m_1+2n_1-3)-M_1(G_1))(n_2(n_1+m_1)-1)^2 -2(n_2(n_1+m_1)-1)M_1(G_2)(M_1(G_1)+EM_1(G_1)+4(m_1+2m_L)) +F(G_2)((n_1+m_1-1)^3 n_1 - F(G_1) - 6(n_1+m_1-1)^2 m_1 +3(n_1+m_1-1)M_1(G_1) + (n_1+m_1-3)^3 m_1 - EF(G_1) -6m_L(n_1+m_1-3)^2 + 3(n_1+m_1-3)EM_1(G_1)).$$

Proof. Using Proposition 3.1, Theorems 3.2 and 2.5 (iii) we get the desired result.

Theorem 4.5 *If G_1 and G_2 are (n_1, m_1) and (n_2, m_2) graphs respectively, then*

$$F(G_1 \otimes_{G^{+-}} G_2) = F(G_2)(m_1^3 n_1 - F(G_1) - 6m_1^3 + 3m_1 M_1(G_1) + (n_1-2)^3 m_1 + EF(G_1) +6m_L(n_1-2)^2 + 3(n_1-2)EM_1(G_1)).$$

Proof. Using the definition of F-index, Propositions 2.6 and 2.7, we obatin desired result.

Corollary 4.6 *If G_1 and G_2 are (n_1, m_1) and (n_2, m_2) graphs respectively, then*

$$F(\overline{G_1 \otimes_{G^{+-}} G_2}) = n_2(n_1+m_1)(n_2(n_1+m_1)-1)^3 -4m_2(2m_1(n_1-3)+M_1(G_1))(n_2(n_1+m_1)-1)^2 +3(n_2(n_1+m_1)-1)M_1(G_2)(M_1(G_1)+EM_1(G_1)+m_1^2(n_1-4) +(n_1-2)(m_1(n_1-2)+4m_L)) - F(G_2)(m_1^3 n_1 - F(G_1) - 6m_1^3 +3m_1 M_1(G_1) + (n_1-2)^3 m_1 + EF(G_1) + 6m_L(n_1-2)^2 +3(n_1-2)EM_1(G_1)).$$

Proof. Using Proposition 3.1, Theorems 3.6 and 2.5 (i) we get the desired result.

Corollary 4.7 *If G_1 and G_2 are (n_1, m_1) and (n_2, m_2) graphs respectively, then*

$$\begin{aligned}\overline{F}(G_1 \otimes_{G^{+-}} G_2) = {} & (n_2(n_1+m_1)-1)M_1(G_2)(M_1(G_1)+EM_1(G_1)+m_1^2(n_1-4)\\ &+(n_1-2)(m_1(n_1-2)+4m_L))-F(G_2)(m_1^3n_1-F(G_1)-6m_1^3\\ &+3m_1M_1(G_1)+(n_1-2)^3m_1+EF(G_1)+6m_L(n_1-2)^2\\ &+3(n_1-2)EM_1(G_1)).\end{aligned}$$

Proof. Using Proposition 3.1, Theorems 3.6 and 2.5 (ii) we get the desired result.

Corollary 4.8 *If G_1 and G_2 are (n_1, m_1) and (n_2, m_2) graphs respectively, then*

$$\begin{aligned}\overline{F}(\overline{G_1 \otimes_{G^{+-}} G_2}) = {} & 2m_2(2m_1(n_1-3)+M_1(G_1))(n_2(n_1+m_1)-1)^2\\ &-2(n_2(n_1+m_1)-1)M_1(G_2)(M_1(G_1)+EM_1(G_1)+m_1^2(n_1-4)\\ &+(n_1-2)(m_1(n_1-2)+4m_L))+F(G_2)(m_1^3n_1-F(G_1)-6m_1^3\\ &+3m_1M_1(G_1)+(n_1-2)^3m_1+EF(G_1)+6m_L(n_1-2)^2\\ &+3(n_1-2)EM_1(G_1)).\end{aligned}$$

Proof. Using Proposition 3.1, Theorems 3.6 and 2.5 (iii) we get the desired result.

Theorem 4.9 *If G_1 and G_2 are (n_1, m_1) and (n_2, m_2) graphs respectively, then*

$$\begin{aligned}F(G_1 \otimes_{\overline{G^{+-}}} G_2) = {} & F(G_2)((n_1-1)^3n_1+F(G_1)+6m_1(n_1-1)^2+3M_1(G_1)(n_1-1)\\ &+(m_1+1)^3m_1-EF(G_1)-6(m_1+1)^2m_L+3EM(G_1)(m_1+1)).\end{aligned}$$

Proof. Using the definition of F-index, Propositions 2.6 and 2.7, we obtain the desired result.

Corollary 4.10 *If G_1 and G_2 are (n_1, m_1) and (n_2, m_2) graphs respectively, then*

$$\begin{aligned}F(\overline{G_1 \otimes_{\overline{G^{+-}}} G_2}) = {} & n_2(n_1+m_1)(n_2(n_1+m_1)-1)^3-4m_2(n_1(n_1-1)+m_1(m_1+5)\\ &-M_1(G_1))(n_2(n_1+m_1)-1)^2+3(n_2(n_1+m_1)-1)M_1(G_2)(M_1(G_1)\\ &+EM_1(G_1)+(n_1-1)^2n_1+(m_1+1)^2m_1+4(m_1(n_1-1)-m_L(m_1+1)))\\ &-F(G_2)(F(G_1)+8m_1+EF(G_1)+24m_L+6EM_1(G_1)).\end{aligned}$$

Proof. Using Proposition 3.1, Theorems 3.10 and 2.5 (i) we get the desired result.

Corollary 4.11 *If G_1 and G_2 are (n_1, m_1) and (n_2, m_2) graphs respectively, then*

$$\begin{aligned}\overline{F}(G_1 \otimes_{\overline{G^{+-}}} G_2) = {} & (n_2(n_1+m_1)-1)M_1(G_2)(M_1(G_1)+EM_1(G_1)+(n_1-1)^2n_1\\ &+(m_1+1)^2m_1+4(m_1(n_1-1)-m_L(m_1+1)))\\ &-F(G_2)(F(G_1)+8m_1+EF(G_1)+24m_L+6EM_1(G_1)).\end{aligned}$$

Proof. Using Proposition 3.1, Theorems 3.10 and 2.5 (ii) we get the desired result.

Corollary 4.12 *If G_1 and G_2 are (n_1, m_1) and (n_2, m_2) graphs respectively, then*

$$\begin{aligned}\overline{F}(\overline{G_1 \otimes_{\overline{G^{+-}}} G_2}) = {} & 2m_2(n_1(n_1-1)+m_1(m_1+5)-M_1(G_1))(n_2(n_1+m_1)-1)^2\\ &-2(n_2(n_1+m_1)-1)M_1(G_2)(M_1(G_1)+EM_1(G_1)+(n_1-1)^2n_1\\ &+(m_1+1)^2m_1+4(m_1(n_1-1)-m_L(m_1+1)))\\ &+F(G_2)(F(G_1)+8m_1+EF(G_1)+24m_L+6EM_1(G_1)).\end{aligned}$$

Proof. Using Proposition 3.1, Theorems 3.10 and 2.5 (iii) we get the desired result.

Theorem 4.13 *If G_1 and G_2 are (n_1, m_1) and (n_2, m_2) graphs respectively, then*

$$\begin{aligned}F(G_1 \otimes_{G^{-+}} G_2) = {} & F(G_2)(F(G_1)+(m_1+1)^3m_1-EF(G_1)-6(m_1+1)^2m_L\\ &+3EM(G_1)(m_1+1)).\end{aligned}$$

Proof. Using the definition of F-index, Propositions 2.6 and 2.7, we obtain the desired result.

Corollary 4.14 *If G_1 and G_2 are (n_1, m_1) and (n_2, m_2) graphs respectively, then*

$$\begin{aligned} F(\overline{G_1 \otimes_{G^{-+}} G_2}) = {} & n_2(n_1+m_1)(n_2(n_1+m_1)-1)^3 \\ & -4m_2(m_1(m_1+5)-M_1(G_1))(n_2(n_1+m_1)-1)^2 \\ & +3(n_2(n_1+m_1)-1)M_1(G_2)(M_1(G_1)+EM_1(G_1)+(m_1+1)^2m_1 \\ & -4m_L(m_1+1))-F(G_2)(F(G_1)+(m_1+1)^3m_1-EF(G_1) \\ & -6(m_1+1)^2m_L+3EM(G_1)(m_1+1)). \end{aligned}$$

Proof. Using Proposition 3.1, Theorems 3.14 and 2.5 (i) we get the desired result.

Corollary 4.15 *If G_1 and G_2 are (n_1, m_1) and (n_2, m_2) graphs respectively, then*

$$\begin{aligned} \overline{F}(G_1 \otimes_{G^{-+}} G_2) = {} & (n_2(n_1+m_1)-1)M_1(G_2)(M_1(G_1)+EM_1(G_1)+(m_1+1)^2m_1 \\ & -4m_L(m_1+1))-F(G_2)(F(G_1)+(m_1+1)^3m_1-EF(G_1) \\ & -6(m_1+1)^2m_L+3EM(G_1)(m_1+1)). \end{aligned}$$

Proof. Using Proposition 3.1, Theorems 3.14 and 2.5 (ii) we get the desired result.

Corollary 4.16 *If G_1 and G_2 are (n_1, m_1) and (n_2, m_2) graphs respectively, then*

$$\begin{aligned} \overline{F}(\overline{G_1 \otimes_{G^{-+}} G_2}) = {} & 2m_2(m_1(m_1+5)-M_1(G_1))(n_2(n_1+m_1)-1)^2 \\ & -2(n_2(n_1+m_1)-1)M_1(G_2)(M_1(G_1)+EM_1(G_1)+(m_1+1)^2m_1 \\ & -4m_L(m_1+1))+F(G_2)(F(G_1)+(m_1+1)^3m_1-EF(G_1) \\ & -6(m_1+1)^2m_L+3EM(G_1)(m_1+1)). \end{aligned}$$

Proof. Using Proposition 3.1, Theorems 3.14 and 2.5 (iii) we get the desired result.

Theorem 4.17 *If G_1 and G_2 are (n_1, m_1) and (n_2, m_2) graphs respectively, then*

$$\begin{aligned} F(G_1 \otimes_{\overline{G^{-+}}} G_2) = {} & F(G_2)((n_1+m_1-1)^3n_1-F(G_1)-6(n_1+m_1-1)^2m_1 \\ & +3(n_1+m_1-1)M_1(G_1)+(n_1-2)^3m_1+EF(G_1)+6m_L(n_1-2)^2 \\ & +3(n_1-2)EM_1(G_1)). \end{aligned}$$

Proof. Using the definition of F-index, Propositions 2.6 and 2.7, we obtain the desired result.

Corollary 4.18 *If G_1 and G_2 are (n_1, m_1) and (n_2, m_2) graphs respectively, then*

$$\begin{aligned} F(\overline{G_1 \otimes_{\overline{G^{-+}}} G_2}) = {} & n_2(n_1+m_1)(n_2(n_1+m_1)-1)^3-4m_2(n_1(n_1-1)+2m_1(n_1-3) \\ & +M_1(G_1))(n_2(n_1+m_1)-1)^2+3(n_2(n_1+m_1)-1)M_1(G_2)(M_1(G_1) \\ & +(n_1+m_1-1)(n_1(n_1+m_1-1)-4m_1)+EM_1(G_1) \\ & +(n_1-2)(m_1(n_1-2)+4m_L))-F(G_2)((n_1+m_1-1)^3n_1-F(G_1) \\ & -6(n_1+m_1-1)^2m_1+3(n_1+m_1-1)M_1(G_1)+(n_1-2)^3m_1 \\ & +EF(G_1)+6m_L(n_1-2)^2+3(n_1-2)EM_1(G_1)). \end{aligned}$$

Proof. Using Proposition 3.1, Theorems 3.18 and 2.5 (i) we get the desired result.

Corollary 4.19 *If G_1 and G_2 are (n_1, m_1) and (n_2, m_2) graphs respectively, then*

$$\begin{aligned} \overline{F}(G_1 \otimes_{\overline{G^{-+}}} G_2) = {} & (n_2(n_1+m_1)-1)M_1(G_2)(M_1(G_1)+(n_1+m_1-1)(n_1(n_1+m_1-1) \\ & -4m_1)+EM_1(G_1)+(n_1-2)(m_1(n_1-2)+4m_L)) \\ & -F(G_2)((n_1+m_1-1)^3n_1-F(G_1)-6(n_1+m_1-1)^2m_1 \\ & +3(n_1+m_1-1)M_1(G_1)+(n_1-2)^3m_1+EF(G_1)+6m_L(n_1-2)^2 \\ & +3(n_1-2)EM_1(G_1)). \end{aligned}$$

Proof. Using Proposition 3.1, Theorems 3.18 and 2.5 (ii) we get the desired result.

Corollary 4.20 *If G_1 and G_2 are (n_1, m_1) and (n_2, m_2) graphs respectively, then*

$$\begin{aligned}\overline{F}(\overline{G_1 \otimes_{\overline{G^{-+}}} G_2}) = {} & 2m_2(n_1(n_1-1)+2m_1(n_1-3)+M_1(G_1))(n_2(n_1+m_1)-1)^2 \\ & -2(n_2(n_1+m_1)-1)M_1(G_2)(M_1(G_1)+(n_1+m_1-1)(n_1(n_1+m_1-1) \\ & -4m_1)+EM_1(G_1)+(n_1-2)(m_1(n_1-2)+4m_L)) \\ & +F(G_2)((n_1+m_1-1)^3 n_1 - F(G_1) - 6(n_1+m_1-1)^2 m_1 \\ & +3(n_1+m_1-1)M_1(G_1)+(n_1-2)^3 m_1 + EF(G_1) + 6m_L(n_1-2)^2 \\ & +3(n_1-2)EM_1(G_1)).\end{aligned}$$

Proof. Using Proposition 3.1, Theorems 3.18 and 2.5 (iii) we get the desired result.

Theorem 4.21 *If G_1 and G_2 are (n_1, m_1) and (n_2, m_2) graphs respectively, then*

$$\begin{aligned}F(G_1 \otimes_{G^{--}} G_2) = {} & F(G_2)(m_1^3 n_1 - F(G_1) - 6m_1^3 + 3m_1 M_1(G_1) + (n_1+m_1-3)^3 m_1 \\ & -EF(G_1) - 6m_L(n_1+m_1-3)^2 + 3(n_1+m_1-3)EM_1(G_1)).\end{aligned}$$

Proof. Using the definition of F-index, Propositions 2.6 and 2.7, we obtain desired result.

Corollary 4.22 *If G_1 and G_2 are (n_1, m_1) and (n_2, m_2) graphs respectively, then*

$$\begin{aligned}F(\overline{G_1 \otimes_{G^{--}} G_2}) = {} & n_2(n_1+m_1)(n_2(n_1+m_1)-1)^3 - 4m_2(m_1(m_1+2n_1-3) \\ & -M_1(G_1))(n_2(n_1+m_1)-1)^2 + 3(n_2(n_1+m_1)-1)M_1(G_2)(M_1(G_1) \\ & +EM_1(G_1)+m_1(n_1 m_1 + (n_1+m_1-3)^2) - 4(m_1^2 + m_L(n_1+m_1-3))) \\ & -F(G_2)(m_1^3 n_1 - F(G_1) - 6m_1^3 + 3m_1 M_1(G_1) + (n_1+m_1-3)^3 m_1 \\ & -EF(G_1) - 6m_L(n_1+m_1-3)^2 + 3(n_1+m_1-3)EM_1(G_1)).\end{aligned}$$

Proof. Using Proposition 3.1, Theorems 3.22 and 2.5 (i) we get the desired result.

Corollary 4.23 *If G_1 and G_2 are (n_1, m_1) and (n_2, m_2) graphs respectively, then*

$$\begin{aligned}\overline{F}(G_1 \otimes_{G^{--}} G_2) = {} & (n_2(n_1+m_1)-1)M_1(G_2)(M_1(G_1)+EM_1(G_1) \\ & +m_1(n_1 m_1 + (n_1+m_1-3)^2) - 4(m_1^2 + m_L(n_1+m_1-3))) \\ & -F(G_2)(m_1^3 n_1 - F(G_1) - 6m_1^3 + 3m_1 M_1(G_1) + (n_1+m_1-3)^3 m_1 \\ & -EF(G_1) - 6m_L(n_1+m_1-3)^2 + 3(n_1+m_1-3)EM_1(G_1)).\end{aligned}$$

Proof. Using Proposition 3.1, Theorems 3.22 and 2.5 (ii) we get the desired result.

Corollary 4.24 *If G_1 and G_2 are (n_1, m_1) and (n_2, m_2) graphs respectively, then*

$$\begin{aligned}\overline{F}(\overline{G_1 \otimes_{G^{--}} G_2}) = {} & 2m_2(m_1(m_1+2n_1-3) - M_1(G_1))(n_2(n_1+m_1)-1)^2 \\ & -2(n_2(n_1+m_1)-1)M_1(G_2)(M_1(G_1)+EM_1(G_1) \\ & +m_1(n_1 m_1 + (n_1+m_1-3)^2) - 4(m_1^2 + m_L(n_1+m_1-3))) \\ & +F(G_2)(m_1^3 n_1 - F(G_1) - 6m_1^3 + 3m_1 M_1(G_1) + (n_1+m_1-3)^3 m_1 \\ & -EF(G_1) - 6m_L(n_1+m_1-3)^2 + 3(n_1+m_1-3)EM_1(G_1)).\end{aligned}$$

Proof. Using Proposition 3.1, Theorems 3.22 and 2.5 (iii) we get the desired result.

Theorem 4.25 *If G_1 and G_2 are (n_1, m_1) and (n_2, m_2) graphs respectively, then*

$$\begin{aligned}F(G_1 \otimes_{\overline{G^{--}}} G_2) = {} & F(G_2)((n_1-1)^3 n_1 + F(G_1) + 6m_1(n_1-1)^2 + 3M_1(G_1)(n_1-1) \\ & +8m_1 + EF(G_1) + 24m_L + 6EM_1(G_1)).\end{aligned}$$

Proof. Using the definition of F-index, Propositions 2.6 and 2.7, we obtain the desired result.

Corollary 4.26 *If G_1 and G_2 are (n_1, m_1) and (n_2, m_2) graphs respectively, then*

$$F(\overline{G_1 \otimes_{\overline{G^{--}}} G_2}) = n_2(n_1+m_1)(n_2(n_1+m_1)-1)^3 - 4m_2(n_1(n_1-1)+2m_1$$

$$
\begin{aligned}
&+M_1(G_1))(n_2(n_1+m_1)-1)^2+3(n_2(n_1+m_1)-1)M_1(G_2)(M_1(G_1)\\
&+EM_1(G_1)+(n_1-1)^2n_1+4(m_1n_1+m_L))\\
&-F(G_2)((n_1-1)^3n_1+F(G_1)+6m_1(n_1-1)^2+3M_1(G_1)(n_1-1)\\
&+8m_1+EF(G_1)+24m_L+6EM_1(G_1)).
\end{aligned}
$$

Proof. Using Proposition 3.1, Theorems 3.26 and 2.5 (i) we get the desired result.

Corollary 4.27 *If G_1 and G_2 are (n_1, m_1) and (n_2, m_2) graphs respectively, then*

$$
\begin{aligned}
\overline{F}(G_1 \otimes_{\overline{G^{--}}} G_2) = {}&(n_2(n_1+m_1)-1)M_1(G_2)(M_1(G_1)+EM_1(G_1)+(n_1-1)^2n_1\\
&+4(m_1n_1+m_L))-F(G_2)((n_1-1)^3n_1+F(G_1)+6m_1(n_1-1)^2\\
&+3M_1(G_1)(n_1-1)+8m_1+EF(G_1)+24m_L+6EM_1(G_1)).
\end{aligned}
$$

Proof. Using Proposition 3.1, Theorems 3.26 and 2.5 (ii) we get the desired result.

Corollary 4.28 *If G_1 and G_2 are (n_1, m_1) and (n_2, m_2) graphs respectively, then*

$$
\begin{aligned}
\overline{F}(\overline{G_1 \otimes_{\overline{G^{--}}} G_2}) = {}&2m_2(n_1(n_1-1)+2m_1+M_1(G_1))(n_2(n_1+m_1)-1)^2\\
&-2(n_2(n_1+m_1)-1)M_1(G_2)(M_1(G_1)+EM_1(G_1)\\
&+(n_1-1)^2n_1+4(m_1n_1+m_L))+F(G_2)((n_1-1)^3n_1+F(G_1)\\
&+6m_1(n_1-1)^2+3M_1(G_1)(n_1-1)+8m_1+EF(G_1)+24m_L\\
&+6EM_1(G_1)).
\end{aligned}
$$

Proof. Using Proposition 3.1, Theorems 3.26 and 2.5 (iii) we get the desired result.

[5] Conclusion

In this paper, we have introduced new tensor products of generalized middle graphs. We obtained explicit formulae for first and second Zagreb indices, F- indices, coindices of these new graphs and their complements.

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