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HUB NUMBER AND HUB-INTEGRITY OF MAP GRAPHS

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ABSTRACT

The hub number of a graph G, a concept related to network concerned problems. A hub set of a graph G is a subset S of vertices of G with the property that for any pair of vertices outside of S, there is a path between them with all intermediate vertices in S. The smallest cardinality of hub set is called *hub number*. Walsh [11] gave hub number of several classes of graphs and also shown that the hub number of any cyclic graph G is at least $g(G) - 3$. Recently, Sultan Mahde et al. [13] introduced the hub-integrity of a graph as a new measures of vulnerability, is defined as, $HI(G) = min\{|S| + m(G - S)\}$, where S is a hub set and $m(G - S)$ is the order of a largest component of $G - S$. In this paper, we obtain hub number and hub-integrity of Dharwad district map, Karnataka state map and India map.

Keywords- hub set, hub number, hub-integrity and map graphs.

[1] INTRODUCTION

A map is a symbolic depiction of a location's specific qualities, often drawn on a flat surface. Maps are visual representations of information about the world. They demonstrate the size and form of nations, the locations of landmarks, and the distances between them in order to teach about the world. Settlement trends, for example, can be depicted on maps. They can pinpoint the exact position of houses and streets in a community.

A map is a symbolic representation of relationships between items, regions, or themes in a given location. Many maps are static, meaning they are set to paper or another lasting media, but others are dynamic or interactive. Maps can represent any location, actual or fictitious, without respect to context or size, as in brain mapping, DNA mapping, or computer network topology mapping. The space being mapped might be two-dimensional, like the earth's surface, three-dimensional, like the earth's interior, or even more abstract spaces of any dimension, like those that result when modelling phenomena with many independent factors.

A map graph is a simple graph generated as the intersection graph of finitely many simply linked and internally disjoint sections of the Euclidean plane in graph theory, a field of mathematics.

In this paper, we have proposed three types of graphs which are Dharwad district map graph, Karnataka state map graph and India map graph. As per the Governement Gazette, we have considered changes up to 2018. In which map graph is defined as below.

Definition 1 *The map graph is a graph, let* $v_1,v_2,...,v_n$ *be the vertices of talukas/districts/*states and if two vertices are adjacent when two talukas/districts/states are sharing their boundaries.

In 2006, Walsh [11] have defined hub number of a graph to study a network related problem. Here, we have proposed a problem in a transport field. A graph G which represents the places/regions in a state or country, with an easy way from one place to another place. The government is considering implementing a rapid-transit system (RTS), and wants to place its stations in places/regions so that to travel between two non-adjacent places/regions, when one can only walk to an adjacent station, take the RTS, and reach to the desired places/regions. The government would like to adapt this plan as cheaply as possible, which involves converting as few places/regions into transit stations. It would helpful to government for creating the taluka/district/capitals of states or country which will be the shortest path from taluka/district/capitals of states or country to concerned region/places. Also this plan reduce expenditures of the government. It wiil be helpful to village peoples go to the taluka/district/capitals of states or country to their respective work from the concerned government offices. It is also save the time and reduce the peoples expenditure. This problem is translated to graph theory as below:

Let G be a graph with vertex set $V(G)$ and S be a subset in a graph G such that $S \subseteq V(G)$ and let $x, y \in V(G)$. An S-path between x and y is a path where all intermediate vertices are from S. A set $S \subseteq V(G)$ is a hub set of G if it has the property that, for any $x, y \in V(G)\backslash S$ there is an S-path in G between x and y. The minimum cardinality of hub set is called *hub number* and is denoted by $h(G)Walsh117-124$. The problem in the previous paragraph can be rephrased as: *what is the smallest size of a hub set in G*?. It is clear that $h(G)$ is well defined for any G, as $V(G)$ is itself a hub set. This inspired us to study hub number of map graphs.

Grauman et al. [7] obtained the relationship between hub number, connected hub number and connected domination number of a graph. Cauresma et al. [5] obtained hub number of join, corona and cartesian product of two connected graphs. Basavanagoud et al. [1, 2] studied hub number of some wheel related graphs and hub number of generalized middle graphs.

Recently, Sultan Mahde et al. [13] have introduced the concept of hub-integrity of a graph as a new measures of vulnerability, is defined as, $HI(G) = min\{|S| + m(G-S)\}$, where S is a hub set and $m(G-S)$ is the order of a largest component of $G-S$. It is denoted as $HI(G)$. Furthermore, Sultan Mahde and Veena Mathad [12] have studied hub-integrity of some operations of graphs.

The present paper is organized as follows: In section 2, we study the preliminaries and propositions of hub number and hub-integrity. In section 3, we obtain the hub number and hub-integrity of Dharwad district map graph and its applications. In section 4, we obtain the hub number and hub-integrity of Karnataka state map graph and its applications. In section 5, we obtain the hub number and hub-integrity of India map graph and its applications.

[2] PRELIMINARIES

In this paper, we have considered nontrivial, connected, simple and undirected graphs. Let G be a graph with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$ and edge set $E(G) = \{e_1, e_2, \ldots, e_m\}$. Thus $|V(G)| = n$ and $|E(G)| = m$ where, n and m are called *order* and *size* of graph G respectively. The pendant vertices are called leaf vertices of graph G, and it is denoted by l. For undefined terminology and notations refer [6, 8].

Proposition 2.1 [11] *The hub number of*

i) The path P_n, $h(P_n) = n - 2$,
ii) The cycle C_n, $h(C_n) = n - 3$,
iii) The complete graph K_n, $h(K_n) = 0$,
iv) The tree T, $h(T) = n - l$.

Proposition 2.2 [13] *The hub-integrity*

i) For complete graph $HI(K_n) = n$,
ii) For path P_n with $n \geq 3$, $HI(P_n) = n - 1$,
iii) For cycle C_n,

$$HI(C_n) = \begin{cases} n-1 & if\ \ n = 4,\ 5; \\ n-2 & if\ \ n \geq 6. \end{cases}$$

iv) For wheel graph $W_{1,n}$,

$$HI(W_{1,n}) = \begin{cases} n, & if\ \ n \leq 4; \\ \left\lceil \frac{n}{3} \right\rceil + 3, & if\ \ n \geq 5. \end{cases}$$

[3] HUB NUMBER AND HUB-INTEGRITY OF DHARWAD DISTRICT MAP GRAPHS

In this section, we have defined Dharwad district map graph and also we have obtain hub number and hub-integrity of Dharwad district map graph and its applications. The below information about Dharwad district is collected from the link
$https://dharwad.nic.in/en/about-district/$.

The administrative headquarters of Dharwad municipality, as well as various educational institutions including Karnataka University, are located in Dharwad district and town, which is in the state of Karnataka. Dharwad, Hubli, Kalghatagi, Navalgund, and Kundgol are the five talukas that make up the district of Dharwad. The HDMC is Karnataka's second largest municipal corporation, behind Bangalore. It combines the twin towns of Hubli, a commercial and industrial powerhouse, and Dharwad, an intellectual and cultural hub. With a total area of 202.3 square kilometres split into 67 wards, the administration is in charge.

The word Dharwad means a resting place in a long journey or a small abode. For centuries, Dharwad served as a gateway between Malenaadu (the western mountains) and bayalu (the plains) and became a getaway for travelers. The name is derived from the Sanskrit words dwarawata, dwara meaning door and wata or wada meaning city.

Definition 2 *The Dharwad district map graph DDM is a graph, let v_1, v_2,...,v_5 be the vertices of talukas of Dharwad district and if two vertices are adjacent when two talukas are sharing their boundaries. The DDM graph is depicted in Fig. 1.*

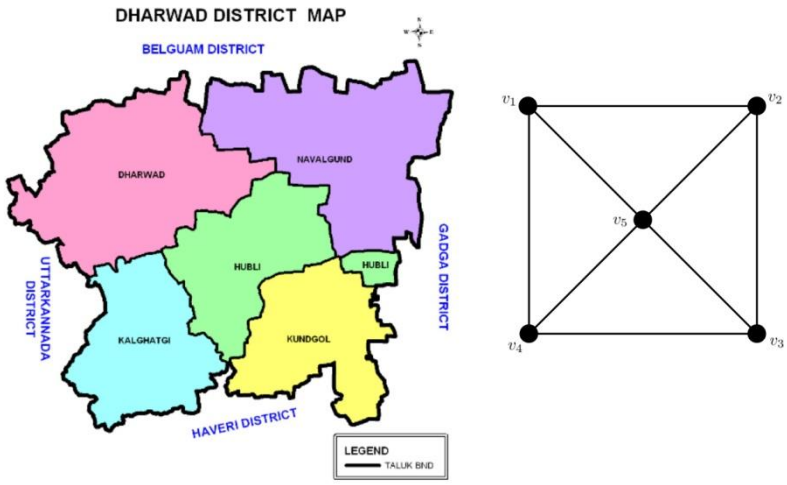

Figure 1: Dharwad district map and its map graph DDM.

Theorem 3.1 *Let DDM be Dharwad district map graph. Then*

$$h(DDM) = 1.$$

Proof. Let v_1, v_2, v_3, v_4, v_5 be vertices of the graph DDM. A Dharwad district map graph $DDM \cong W_{1,4}$. Therefore, $h(DDM) = 1$ (since $h(W_{1,n}) = 1$ [1]).

Theorem 3.2 *Let DDM be Dharwad district map graph. Then*

$$HI(DDM) = 4.$$

Proof. Let v_1, v_2, v_3, v_4, v_5 be vertices of the graph DDM. A Dharwad district map graph $DDM \cong W_{1,4}$. Therefore, $HI(DDM) = 4$ (since $HI(W_{1,n}) = n$ [13]).

3.1 Applications

If Karnataka Government implement this plan, i.e., Hubli has the district administrative offices, then the following benefits are applicable to Dharwad district peoples.

- Presently, Hubli is the hub for the north Karnataka.
- Short distance from Hubli to all other talukas.
- Time saving and reduce the cost of transportation.
- It will helpful to give better administration and governance.
- Easily transport materials from district administrative offices to concerned taluka offices, like, ration, medical things, etc.
- Peoples of Dharwad district can visit district offices for their work in less time and less cost.

- District offices can reduce their expenditures like, Government vehicles petrol/diesel charges.
- Higher officers can visit and inspection taluka offices in time with less cost and less time.
- Employees can easily reach their office in time.

[4] HUB NUMBER AND HUB-INTEGRITY OF KARNATAKA STATE MAP GRAPHS

In this section, we have defined Karnataka state map graph and also we have obtain hub number and hub-integrity of Karnataka state map graph and its applications. The below information about Karnataka state is collected from the link $https://www.karnataka.com/profile/about-profile/.$

Karnataka is situated in the south western region of India. It is one of the most prosperous states in India. Karnataka has made tremendous progress in the fields of education, industry, agriculture, literature and tourism. Bengaluru is the capital of Karnataka. Bengaluru is known as the Silicon Valley of Asia due to its flourishing information technology industry.

Karnataka was formed on 1^{st} November 1956 when the States Reorganisation Act came into effect. At that time Karnataka was known as the State of Mysore. It was renamed Karnataka in the year 1973. Every year 1^{st} November is celebrated as the Kannada Rajyotsava (Karnataka formation day). It is done to commemorate the merger of all the Kannada language-speaking areas of South India into a single state. The day is a significant part of Karnataka's history.

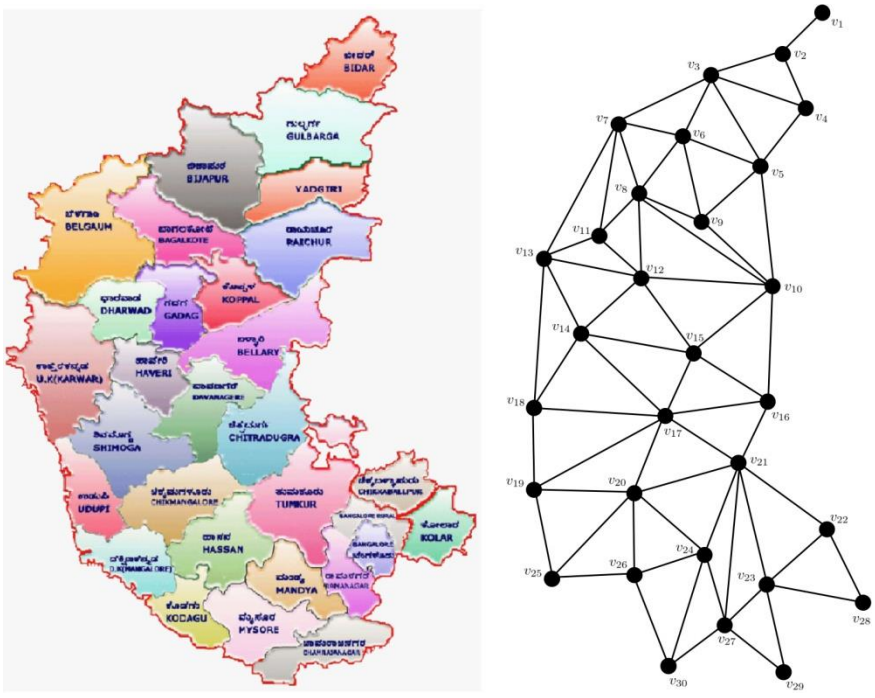

Figure 2: Karnataka state map and its map graph KSM.

Karnataka is the seventh largest state in India in terms of area. It has 30 districts. Karnataka is bounded by the Arabian Sea and the Laccadive Sea on the west, Goa on the north-

west, Maharashtra on the north, Telangana on the north-east, Andhra Pradesh on the east, Tamil Nadu on the south-east and Kerala on the south-west.

Definition 3 *The Karnataka state map graph KSM is a graph, let v_1, v_2,...,v_{30} be the vertices of districts of Karnataka and if two vertices are adjacent when two districts are sharing their boundaries. The KSM graph is depicted in Fig. 2.*

Theorem 4.1 *Let KSM be Karnataka state map graph. Then*

$$h(KSM) = 11.$$

Proof. Let $v_1, v_2, v_3, \ldots, v_{30}$ be vertices of the graph KSM. Let $S \subset V(KSM)$. Choose a set $S = \{v_2, v_3, v_6, v_8, v_{12}, v_{14}, v_{17}, v_{20}, v_{21}, v_{23}, v_{24}\}$ which forms an S-path in KSM. So, $|S| = 11$. For any two vertices $x, y \in V(KSM)\backslash S$, there exist an S-path between them in KSM. Therefore S is a hub set of KSM gives the minimum cardinality. Hence, $h(KSM) = 11$.

Theorem 4.2 *Let G be Karnataka State map KSM graph. Then*

$$HI(KSM) = 17.$$

Proof. Let $v_1, v_2, v_3, \ldots, v_{30}$ be vertices of the graph KSM. Let $S \subset V(KSM)$. Choose a set $S = \{v_2, v_3, v_6, v_8, v_{10}, v_{12}, v_{14}, v_{17}, v_{18}, v_{20}, v_{21}, v_{23}, v_{24}, v_{26}\}$ which forms required S-path in KSM. If we remove these vertices in KSM, then we obtain disconnected graph with maximum 3 components. Thus, $|S| = 14$ and $m(G - S) = 3$. Therefore, $HI(KSM) = 17$.

4.1 Applications

If Karnataka Government consider as Davangere is the capital of Karnataka, then the following benefits are applicable to citizens of Karnataka.

- Davanagere has the less distance from all other districts.
- Presently Bengaluru is the hub for the Karnataka.
- Davanagere is also called the heart of Karnataka state.
- Present capital city Bengaluru has many industries and heavy traffic, if implement this plan, will control the traffic and also gives the opportunity to develop other districts of Karnataka.
- It will helpful to host of reputed multinational industrial and research organisations.
- Peoples of Karnataka state can visit capital city offices for their work in less time and less cost.
- In every ten years, capital city will change to other cities of Karnataka. Then it would be helpful to create good infrastructure in all other states of cities, also MNC companies will invest money for their business in other districts, traffic can reduce, and many more.

[5] HUB NUMBER AND HUB-INTEGRITY OF INDIA MAP GRAPHS

In this section, we have defined India map graph and also we have obtain hub number and hub-integrity of India map graph and its applications. In India map, we have considered all states of India and Delhi, as on 2018 previous gazette formed by Government of India. The below information about India is collected from the link
$https://en.wikipedia.org/wiki/India.$

India is a country in south Asia that is formally known as the Republic of India. It is the world's seventh-largest country in terms of land area, the second-most populated country, and the world's most populous democracy. It shares land borders with Pakistan to the west, China, Nepal, and Bhutan to the north, and Bangladesh and Myanmar to the east, all of which are bounded by the Indian Ocean on the south, the Arabian Sea on the southwest, and the Bay of Bengal on the southeast. India borders Sri Lanka and the Maldives in the Indian Ocean, and the Andaman and Nicobar Islands have a maritime border with Thailand, Myanmar, and Indonesia.

Definition 4 *The India map graph IM is a graph, let v_1, v_2,...,v_{30} be the vertices of states of India and if two vertices are adjacent when two states are sharing their boundaries. The IM graph is depicted in Fig. 3.*

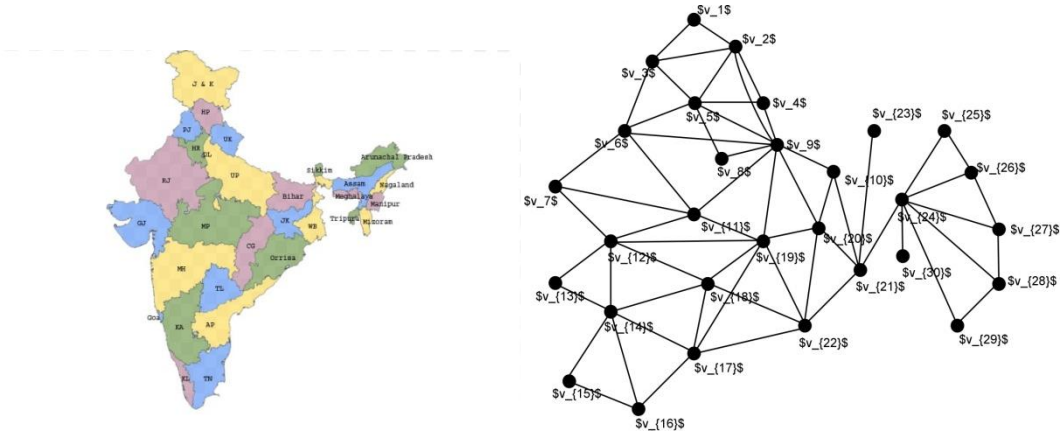

Figure 3: India map and its map graph IM.

Theorem 5.1 *Let IM be India map graph. Then*

$$h(IM) = 9.$$

Proof. Let $v_1, v_2, v_3, \ldots, v_{30}$ be vertices of the graph IM. Let $S \subset V(IM)$. Choose a set $S = \{v_3, v_5, v_9, v_{11}, v_{12}, v_{14}, v_{20}, v_{21}, v_{24}\}$ which forms an S-path in IM. So, $|S| = 9$. For any two vertices $x, y \in V(IM)\backslash S$, there exist an S-path between them in IM. Therefore S is a hub set of IM gives the minimum cardinality. Hence, $h(IM) = 9$.

Theorem 5.2 *Let IM be India map graph. Then*

$$HI(IM) = 14.$$

Proof. Let $v_1, v_2, v_3, \ldots, v_{30}$ be vertices of the graph IM. Let $S \subset V(IM)$. Choose a set $S = \{v_3, v_5, v_9, v_{11}, v_{12}, v_{14}, v_{17}, v_{20}, v_{21}, v_{24}, v_{27}\}$ which forms an S-path in IM. If we remove these vertices in IM, then we obtain disconnected graph with maximum three components. Thus, $|S| = 11$ and $m(G - S) = 3$. Therefore, $HI(IM) = 14$.

5.1 Applications

If Government of India consider as Madhya Pradesh is the capital of India, then the following benefits are applicable to citizens of India.

- Bhopal (Madhya Pradesh) is nearest to all other states of India.
- Present capital city Delhi has many industries and heavy traffic, if implement this plan, will control the traffic and also gives the opportunity to develop other states of cities.
- Bhopal is very much nearer to southern states compared to Delhi. It is well connected by rail, road and air.
- In every ten years, capital city will change to other cities of India. Then it would be helpful to create good infrastructure in all other states of cities, also MNC companies will invest money for their business in other cities, traffic can reduce, and many more.

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