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REVIEW PAPER ON IMAGINARY NUMBERS IN ELECTRICAL ENGINEERING

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ABSTRACT:

The concept of imaginary numbers, particularly encapsulated by *unit i* ***, represents a pivotal advancement in mathematical thought, challenging traditional notions of numerical reality. Initially perceived as a paradoxical entity due to its definition as the square root of -1. This paper delves into the origins and evolution of 'i', tracing its roots from the early musings of mathematicians grappling with unsolvable equations to its formal integration within the framework of complex numbers. The mathematical community's acceptance of 'i' as a legitimate and practical entity marked a paradigm shift, enabling solutions to previously intractable problems in fields such as electrical engineering, quantum mechanics, and signal processing.***

Keywords: *Electrical Engineering, Imaginary numbers.*

[1] INTRODUCTION: IMAGINARY NUMBERS

A Greek mathematician called Hero of Alexandria first invented imaginary numbers. Later in 1572, an Italian mathematician Gerolamo Cardano developed the rules for multiplying imaginary numbers. These numbers help find the square roots of negative numbers. **Imaginary numbers** are numbers that result in a negative number when squared. They are also defined as the square root of negative numbers. An imaginary number is the product of a non-zero real number and the imaginary unit **"i"** (which is also known as **"iota"**),

where $i = \sqrt{(-1)}$ (or) $i^2 = -1$.

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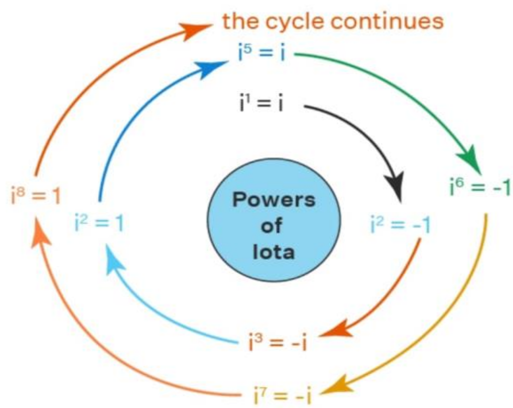

- **Multiplying Imaginary Numbers**

We multiply the imaginary numbers just like how we multiply the terms in algebra. Here, we may have to use the rule of exponents am × an = am+n. But here, we have to take care of the fact that i2 = -1. Here are some examples.

2i × 3i = 6i2 = 6(-1) = -6

3i2 × -5i3 = -15i5 = -15 (i2)2 i = -15 (-1)2 i = -15i

Simplifying the powers of iota may be difficult. Here are some rules that make the process of finding powers of "i" easier while multiplying complex numbers.

i4k = 1

i4k+1 = 1

i4k+2 = -1

i4k+3 = -i

For this complex number, a + jb is called the rectangular form, while r ∠θ is called the polar form. If a complex number is given in polar form, its rectangular form can be found:

$$a = r\cos\theta$$
$$b = r\sin\theta$$

[2] Complex Numbers in Electrical Engineering

Working with Complex Impedance

Voltage and current are always real, observable quantities. In a linear A/C circuit with a sinusoidal stimulus, they will always have a form like $V(t) = V_0 \cos(\omega t + \phi)$. The algebraic complexities come in when we introduce capacitors and inductors, which produce $\pm 90^\circ$ changes in phase. Adding sines and cosines with differing phases is algebraically painful, requiring expertise with trig identities. However, if the circuit is described by linear differential equations, then we can simplify life by adding an imaginary part to the voltage or current:

$$V(t) = V_0 \cos(\omega t + \phi) + j \cdot V_0 \sin(\omega t + \phi) = V_0 e^{j \cdot (\omega t + \phi)}$$

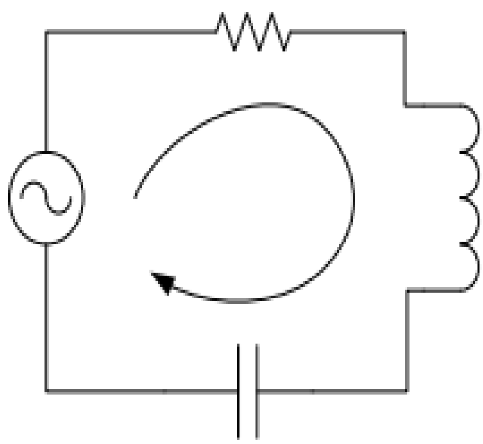

Figure 3. LRC series circuit.

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with the understanding that the observed voltage is just the real part of this expression. Now, when you do your circuit analysis you get to deal with the simple properties of the exponential function instead of nasty trig identities. When done, just take the real part of the final result, and that is your answer. As you will see, what this procedure will do for you is turn a set of linear differential equations into a set of linear algebraic equations.[3]

This works *only because the circuit is a linear circuit, described by linear differential equations.* Since linear equations do not involve any squares, square roots, and so forth of the voltage or current, or multiplication of one voltage or current by another, the real and imaginary parts don't get mixed up. Take a look at the equations in the previous section. The addition and subtraction equations do not mix up the real and imaginary parts, but the equations for multiplication and division do. Multiplying a complex number by a real constant also obviously does not mix up the real and imaginary parts. Essentially, a linear equation is one that will not mix up the real and imaginary parts of the voltages and currents. From a practical standpoint, a linear circuit is one that includes only passive components (resistors, capacitors, and inductors) plus voltage and/or current sources. No diodes, transistors, vacuum tubes, etc. are allowed.

It is perhaps worth mentioning here that the same formalism, with the same advantages of using complex numbers, works in mechanics when dealing with small, harmonic oscillations of mechanical systems.

The recipe for obtaining the steady-state[4] harmonic response of a linear circuit is straightforward. Write each non-static voltage or current source as a complex number:

$$V_0 e^{j\phi} \text{ or } I_0 e^{j\phi}$$

where the phase ϕ can be taken to be zero if there is only one source. Otherwise the relative phases of the sources must be taken into account. Then treat each passive component as an impedance

$$\text{Resistor: } Z = R$$

$$\text{Capacitor: } Z = \frac{1}{j\omega C}$$

$$\text{Inductor: } Z = j\omega L$$

where in general the impedance relates the voltage *across* a component to the current passing *through* the component according to a generalization of Ohm's law:

$$V = IZ$$

Use Kirchhoff's laws to write a set of linear equations for the currents and voltage in the circuit, exactly as you would do for a circuit made up of batteries and resistors. The only difference is that some of the "resistances" are imaginary, so what you end up with is a set of complex linear equations. Solve the equations for the currents and voltages. This is tedious to do by hand, but keep in mind that a computer can solve an amazingly large set of complex linear equations in an instant, using standard "canned" programs. Many scientific calculators also have built-in functions for solving sets of linear complex equations. Finally, express the resulting voltages and/or currents in polar form, from which you can read off the amplitude and phase of each current or voltage.

As an example not included in Horowitz and Hill, let's analyze the standard series LRC circuit (Figure 3) which has a voltage oscillator in series with a resistor, capacitor, and inductor. The differential equation for this circuit follows from adding up the voltage changes around the loop:

$$V_0 e^{j\omega t} - IR - L\frac{dI}{dt} - \frac{Q}{C} = 0,$$

where $V_0 e^{j\omega t}$ is the driving voltage, expressed as a complex quantity as suggested above, with an assumed phase $\phi = 0$. Using $Q = \int I dt$, we get an equation for the current:

$$L\frac{dI}{dt} + \frac{1}{C}\int I dt + RI = V_0 e^{j\omega t}.$$

This is readily solved by making the substitution $I = I_0 e^{j(\omega t + \phi)}$, which turns the differential equation into an algebraic equation:

$$\left(j\omega L + \frac{1}{j\omega C} + R \right) \cdot I_0 e^{j\phi} = V_0.$$

The quantity in parentheses is exactly the "impedance" that one would get by using the impedance rules listed above for resistors, capacitors, and inductors, plus the rule that impedances in series simply add up. So, from now on do not bother to write down the differential equation! Just assume the rules for complex impedance and immediately write down the algebraic equation.

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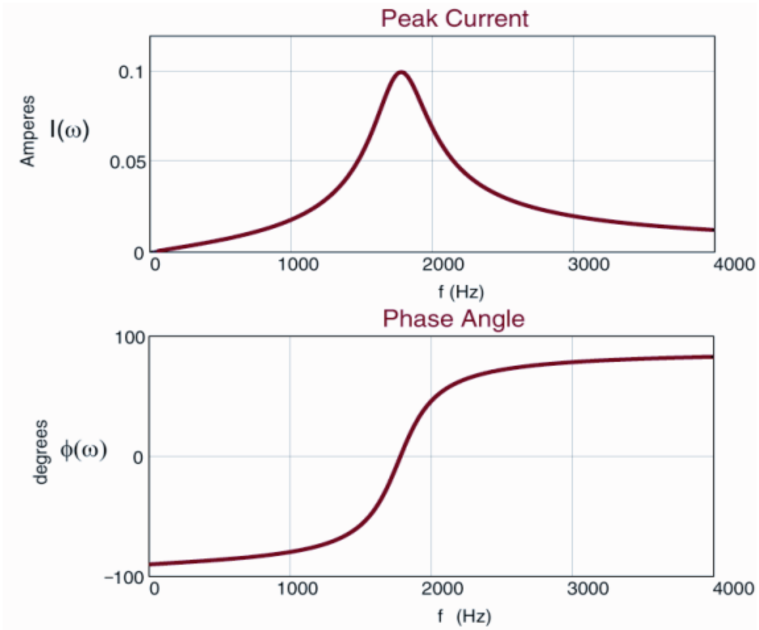

Figure 4. Resonance curves for an LRC series circuit, with R=10 Ohms, C=2PF, and L=4mH

To analyze the series LRC circuit without writing any differential equation, we start with "Ohm's Law" for a reactive circuit:

$$I = \frac{V_0}{Z} \text{ with } Z = R + \frac{1}{j\omega C} + j\omega L\,.$$

To do the division, I convert the impedance to polar form:

$$Z = R + j\cdot\left(\omega L - \frac{1}{\omega C}\right) = \sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}\cdot e^{j\phi_z}$$

with $\phi_Z = \arctan\left(\frac{\omega L - \frac{1}{\omega C}}{R}\right) = \arctan\left(\frac{\omega^2 - \omega_0^2}{\gamma\omega}\right)$ and $\omega_0 \equiv \frac{1}{\sqrt{LC}}$ and $\gamma \equiv \frac{R}{L}$.

So the current is given by

$$I = \frac{V_0}{\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}}\cdot e^{-j\phi_z} = \frac{V_0\cdot\frac{\omega}{L}}{\sqrt{\gamma^2\omega^2 + \left(\omega^2 - \omega_0^2\right)^2}}\cdot e^{j\phi}$$

with $\phi = -\arctan\left(\dfrac{\omega^2 - \omega_0^2}{\gamma\omega}\right)$ for the phase of the current.

This result exhibits a resonance, with ω_0, the natural frequency of the circuit, being the frequency at which the impedance is minimum (and equal simply to R) and the current is maximum, with a phase shift of zero relative to the voltage. Also, γ is a measure of the amount of damping in the circuit and, thus, the width of the resonance curve. This resonance behavior is illustrated in Figure 4.

Analyzing a More Complex Linear Circuit

A more complicated looking example is shown in Figure 5, where the driving voltage is the real part of $V(t) = 10e^{i\omega t}$ volts, with angular frequency $\omega = 10^4$ radians/s. The impedance of the inductor is $j\omega L = 4j$ ohms, and the impedance of the capacitor is $1/j\omega C = -0.25j$ ohms. The objective is to find all the currents in the circuit and the equivalent impedance of the overall circuit, as seen by the voltage source. In this case there are 4 loops, so we will have 4 loop equations and 3 node equations. This goes beyond the complexity that you will see in homework or on any exam, but I throw it in as a random demonstration that the analysis is straightforward and can be formulated in a manner that makes a solution by computer fairly easy.

I prefer to work with the concept of "loop currents," in order to avoid having to write down the node equations. To understand this concept, look at the circuit as redrawn in Figure 6. The four loops are evident, and each is associated with a loop current. The current through the capacitor is clearly i_4, the current through the voltage source is i_1, and the current through the 2-ohm resistor is i_3. However, each of the other 4 components has two currents flowing through it. For example, the current flowing upward through the inductor is $i_3 - i_2$, and the current flowing downward through the leftmost resistor is $i_1 - i_2$. Now, let's apply Kirchhoff's loop law to loop #1, starting at the lower left corner and proceeding upwards through the voltage source, in the direction of loop current i_1:

$$10 - (i_1 - i_2) \cdot 1 = 0$$

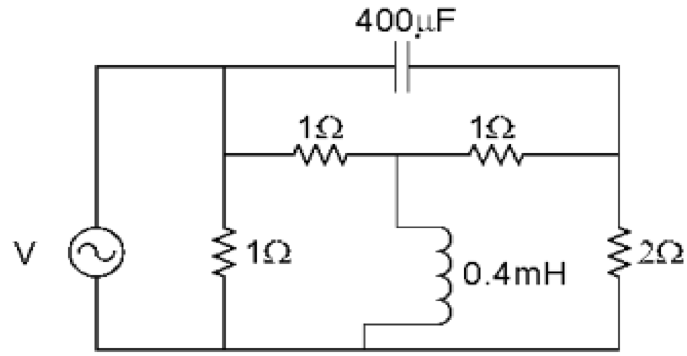

Figure 5. Example of a 4-loop linear circuit.

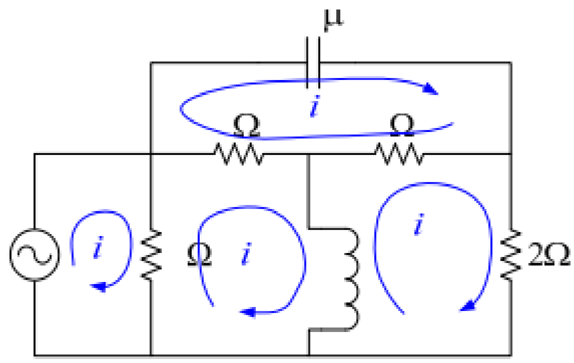

Figure 6. The circuit redrawn with loop currents.

Do the same for loop #2, starting in the lower left hand corner and proceeding upwards through the 1-ohm resistor, in the direction of the loop current i_2:

$$-(i_2 - i_1)\cdot 1 - (i_2 - i_4)\cdot 1 - (i_2 - i_3)\cdot 4j = 0$$

The other two equations, for loops 3 and 4 respectively, are

$$-(i_3 - i_2)\cdot 4j - (i_3 - i_4)\cdot 1 - i_3 \cdot 2 = 0$$

$$i_4 \cdot 0.25j - (i_4 - i_3)\cdot 1 - (i_4 - i_2)\cdot 1 = 0$$

Such equations are easiest to deal with if organized in matrix notation:

$$\begin{pmatrix} 1 & -1 & 0 & 0 \\ 1 & -2-4j & 4j & 1 \\ 0 & 4j & -3-4j & 1 \\ 0 & 1 & 1 & -2+0.25j \end{pmatrix} \times \begin{pmatrix} i_1 \\ i_2 \\ i_3 \\ i_4 \end{pmatrix} = \begin{pmatrix} 10 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Solving these equations by hand would be tedious and annoying, but doing it by computer with a program like Mathcad, Mathematica, or Matlab couldn't be easier. For example, in Mathcad let's call the matrix Z, so the equation looks like

$$Z \cdot I = V$$

Fill the 16 complex values into the matrix Z and the 4 values into V, and then type

$$I = Z^{-1} \cdot V$$

and you're done![5] The result is

$$I = \begin{pmatrix} 15.457 - 1.787j \\ 5.457 - 1.787j \\ 4.990 + 0.652j \\ 5.213 + 0.084j \end{pmatrix}$$

Here is how to interpret the result. For example, the current i_1 can be written in polar form as $i_1 = 15.56e^{-j0.037\pi}$, so the current as a function of time is

$$i_1(t) = 15.56 \cdot \cos(\omega t - 0.037\pi).$$

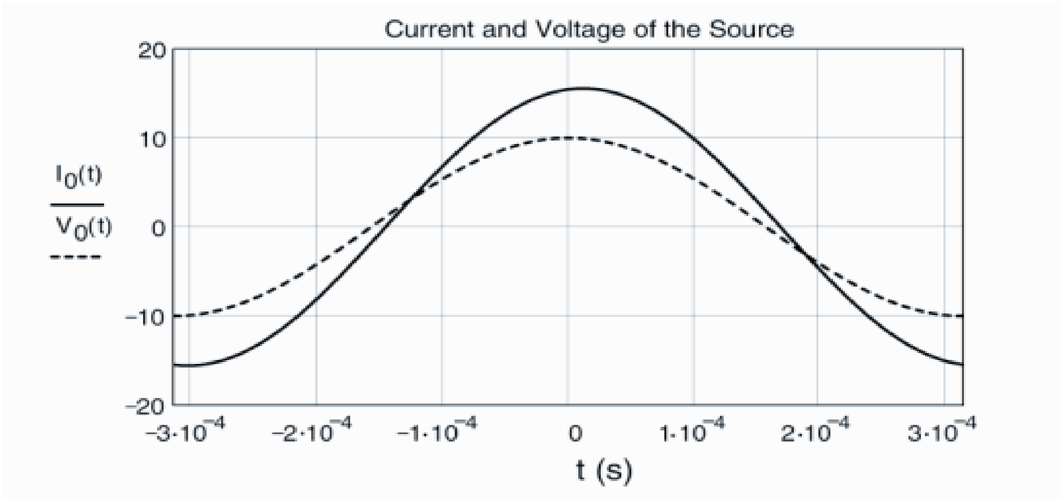

Figure 7. Plots of the voltage and current of the voltage supply as a function of time for a supply frequency of 10^4 radians/s. The current lags behind the voltage by several degrees.

That is, the current passing through the source lags behind the voltage by 0.037π radians, or about 6.7 degrees. Figure 7 shows how the current and voltage would look if displayed on an oscilloscope. The equivalent impedance of the circuit, as seen by the source, can be calculated from the ratio of the voltage and current of the source:

$$Z_{eq} = \frac{V}{i_1} = \frac{10}{15.56} \cdot e^{+j0.037\pi}.$$

Thus at this frequency, the circuit looks slightly inductive to the source.

CONCLUSION

In summary, complex numbers provide a straightforward and powerful mathematical framework for analyzing linear circuits, enabling engineers to predict and optimize circuit behavior accurately. This understanding is foundational for advanced studies and practical applications in electrical engineering and physics.

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