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FOURTH ORDER LAPLACE TRANSFORM IN BICOMPLEX SPACE WITH APPLICATION IN PERIODIC FUNCTION

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ABSTRACT

In this paper we will evaluate fourth order Laplace transform of Periodic function. Bicomplex numbers are pairs of complex numbers with commutative ring with unity and zero-divisors which describe physical interpretation in four dimensional space and provide large class of frequency domain.

Keywords - Bicomplex numbers, Fourth order Laplace Transform.

[1] INTRODUCTION

A lot of research have been done, in past few years in the applications of bicomplex functions.The concept of bicomplex numbers was introduced by Serge[6] in order to compactly describe physical interpretation in four-dimensional space. In fact, bicomplex numbers are generalization of complex numbers with some non-invertible elements interpolated on the null cone.

Recently, enormous efforts have been done to expand the theory of integral transforms in bicomplex space and studied their applications by *Agarwal* et al. [16, 17, 18, 20]. However, in recent bicomplex Schödinger equation and some of its properties studied by *Rochon* and *Trembley* [8] and self adjoint operators were defined for finite and infinite dimensional bicomplex Hilbert spces [9, 23, 24]. An analytical method to solve bicomplex version of Schödinger equation corresponds to the Hamiltonian system was studied by *Banerjee* [2,3]. *Lavoie* et al [25] examined the quantum harmonic oscillator problem in bicomplex numbers and obtained eigenvalues and eigenkets of the bicomplex harmonic oscillator. *Kumar* et al.

[22] introduced the bicomplex version of topological vector spaces and topological modules were developed by *Kumar* and *Saini* [21] over the ring of bicomplex numbers. *Cerejeiras* et al, [14] reconstructed a bicomplex sparse signal with high probability from a reduced number of bicomplex random samples. *Ghanmi and* Zine [4] introduced bicomplex Segal-Bargmann and fractional Fourier transforms.

Double Laplace transform proposed by *Van der Pol* [27] an applied by *Humbert* [15] in the study of hypergeometric functions; by *Jaeger* [11] to solve boundary value problems in heat conduction. The complex double Laplace transform was expanded to multiple Laplace transform in *n* independent complex variables by *Estrin* and *Higgins* [26]. Applications of triple Laplace transform in solving third order patial differential Mboctara equation was discussed by *Atangana* [1]. *Agarwal* et al. [19] generalized double Laplace transform to bicomplex double Laplace transform and found some applications.

For solving the large class of partial differential equations of bicomplex-valued function, we require integral transforms defined for large class. In this procedure we derive triple Laplace fined for large class. In this procedure we derive triple Laplace transform in bicomplex space with ROC that can be competent the transferring signals from real-valued *(x , y, z,t)* domain to bicomplexified frequency $(\xi, \eta, \gamma, \kappa)$ domain.

[2] PRELIMINARIES OF BICOMPLEX NUMBERS

The set of complex numbers C which is ordered pair of two real numbers in complex plane with a non – real unit i_1 such that $i_1^2 = -1$, represented as

$C = \{z = x + iy: x, y \in R\}$ ….. (1)

where R is the set of real numbers.

Similarly, the set of bicomplex numbers C_2 which is ordered pair of two complex numbers with non real units i_1 and i_2 such that

$i_1^2 = i_2^2 = -1, i_1 i_2 = i_2 i_1 = j$; $j^2 = 1$, represented as

$C_2 = \{\xi = z_1 + i_2 z_2 : z_1, z_2 \epsilon C\}$…….. (2)

Or

$C_2 = \{\xi = x_0 + i_1 x_1 + i_2 x_2 + j x_3 : x_0, x_1, x_2, x_3 \epsilon R\}$…… (3)

Bicomplex numbers can be represented using idempotent elements $e_1 = \frac{1+i_1 i_2}{2}$ and $e_2 = \frac{1-i_1 i_2}{2}$ with $e_1 + e_2 = 1$ and $e_1 e_2 = e_2 e_1 = 0$. $In\ fact\ for\ every\ \xi = z_1 + i_2 z_2 \epsilon C_2$, we get

$$z_1 + i_2 z_2 = (z_1 - i_1 z_2)e_1 + (z_1 + i_1 z_2)e_2$$

$$= P_1(\xi)e_1 + P_2(\xi)e_2$$

where the projections $P_1 : C_2 \rightarrow C$ and $P_2 : C_2 \rightarrow C$ are defined as

$$P_1(z_1 + i_2 z_2) = z_1 - i_1 z_2$$

and $P_2(z_1 + i_2 z_2) = z_1 + i_1 z_2$ respectively.

$\{e_1, e_2\}$ in idempotent basis of bicomplex numbers. All details of bicomplex holomorphic functions and bicomplex numbers. Can be seen from [7 , 10 , 13].

[3] BICOMPLEX FOURTH ORDER LAPLACE TRANSFORM

Let $f(x, y, z, t)$ be a bicomplex valued function of few variable $x, y, z, t > 0$ which is piecewise continuous and has exponential order k_1, k_2, k_3 and k_4 respectively.

The bicomplex Laplace transform (see Kumar and Kumar [5]) w.r.t. x is

$L_x[f(x, y, z, t); \xi] = \int_0^\infty e^{-\xi x} f(x, y, z, t)dx$

$= \bar{f}(\xi, y, z, t), \ \xi \epsilon \Omega_1 \subset C_2$ …… (4)

where $\Omega_1 = \{\xi = \xi_1 e_1 + \xi_2 e_2 \epsilon C_2 : Re(P_1 : \xi) > k_1 \ and \ Re(P_2 : \xi) > k_1$…..(5)

Or $\Omega_1 = \{\xi \epsilon C_2 : Re(\xi) > k_1 + |Imj\ (\xi)|\}$ ….(6)

where $Imj\ (\xi)$ denotes the imaginary part of ξ w.r.t. j.

The integral in (4) is convergent and bicompex holomorphic in Ω_1.

Similarly, bicomplex Laplace transform of $f(x, y, z, t)$ w.r.t. y is

$L_y[f(x, y, z, t); \eta] = \int_0^\infty e^{-\eta y} f(x, y, z, t)dy$

$= \bar{f}(x, \eta, z, t), \ \eta \epsilon \Omega_2 \subset C_2$ ……(7)

where $\Omega_2 = \{\eta = \eta_1 e_1 + \eta_2 e_2 \epsilon C_2 : Re(P_1 : \eta) > k_2 \ and \ Re(P_2 : \eta) > k_2$…..(8)

Or $\Omega_2 = \{\eta \epsilon C_2 : Re(\eta) > k_2 + |Imj\ (\eta)|\}$ ….(9)

where (7) is convergent and bicomplex holomorphic in Ω_2.

Bicomplex Laplace transform of $f(x, y, z, t)$ w.r.t. z is

$L_z[f(x, y, z, t); \gamma] = \int_0^\infty e^{-\gamma z} f(x, y, z, t)dz$

$= \bar{f}(x, y, \gamma, t), \ \gamma \epsilon \Omega_3 \subset C_2$ ……(10)

where $\Omega_3 = \{\gamma = \gamma_1 e_1 + \gamma_2 e_2 \epsilon C_2 : Re(P_1 : \gamma) > k_3 \ and \ Re(P_2 : \gamma) > k_3$…..(11)

Or $\Omega_3 = \{\gamma \epsilon C_2 : Re(\gamma) > k_3 + |Imj\ (\gamma)|\}$ ….(12)

and also bicomplex Laplace transform of $f(x,y,z,t)$ w.r.t. t is

$L_t[f(x,y,z,t);\kappa] = \int_0^\infty e^{-\kappa t} f(x,y,z,t)dt$

$= \bar{f}(x,y,\gamma,\kappa),\ \ \kappa \in \Omega_4 \subset\ C_2$ ……(13)

where $\Omega_4 = \{\kappa = \kappa_1 e_1 + \kappa_2 e_2 \epsilon C_2 : Re(P_1 : \kappa) > k_4\ and\ Re(P_2 : \kappa) > k_4$…..(14)

Or $\Omega_4 = \{\kappa \epsilon C_2 : Re(\kappa) > k_4 + |Imj\ (\kappa)|\}$ ….(15)

Now, taking the bicomplex Laplace transform of (4) w.r.t. y, z *and* t *using (7) , (10) and (13), we have*

$$L_{x,y,z,t}[f(x,y,z,t);\xi,\eta,\gamma,\kappa] = \int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty e^{-(\xi x+\eta y+\gamma z+\kappa t)} f(x,y,z,t)dxdydzdt$$

$= \bar{\bar{\bar{f}}}(\xi,\eta,\gamma,\kappa),\ \ (\xi,\eta,\gamma,\kappa)\epsilon\Omega$….(16)

And the integral on right hand side is convergent and bicomplex holomorphic in

$\Omega = \{(\xi,\eta,\gamma,\kappa)\epsilon C_2^4;\ \xi\epsilon\Omega_1,\ \eta\epsilon\Omega_2, \gamma\epsilon\Omega_3\ and\ \kappa\epsilon\Omega_4\}$….(17)

Now we will define the bicomplex fourth order Laplace transform as**:**

Definition

Let $f(x,y,z,t)$ be a bicomplex-valued function of four variables $x,y,z,t > 0$, which is piecewise continuous and has exponential order $\kappa_1,\kappa_2,\kappa_3$ and κ_4 with respect to x,y,z and t respectively. We say the transform in (16) as bicomplex fourth order Laplace transform.

[4] MAIN RESULT

Fourth order Laplace Transform of Periodic function

$$L_{x,y,z,t}[f(x,y,z,t)] = \frac{\int_0^{T_1}\int_0^{T_2}\int_0^{T_3}\int_0^{T_4} e^{-(\xi x+\eta y+\gamma z+\kappa t)} f(x,y,z,t)dxdydzdt}{(1-e^{-\zeta T_1})(1-e^{-\eta T_2})(1-e^{-\gamma T_3})(1-e^{-\kappa T_4})}$$

$\mathrm{Re}(\xi) > |Imj(\xi)|$, $\mathrm{Re}(\eta) > |Imj(\eta)|$, $\mathrm{Re}(\gamma) > |Imj(\gamma)|$ and $\mathrm{Re}(\kappa) > |Imj(\kappa)|$

Proof:

Let $f(x,y,z,t)$ be a periodic function with period T_1 with respect to x . The for $\xi \in c_2$ and $\mathrm{Re}(\xi) > |Imj(\xi)|$ see Agarwal et al. [20]

$$L_x[f(x,y,z,t)] = \frac{\int_0^{T_1} e^{-\xi x} f(x,y,z,t)dx}{(1-e^{-\zeta T_1})} = \bar{f}(\xi,y,z,t) \; \ldots .(18)$$

Similarly, for η ∈ c_2 and Re$(\eta) > |Imj(\eta)|$ taking the bi complex Laplace Transform of (18) with respect to y, we have

$$L_y[\bar{f}(\xi,y,z,t)] = \bar{\bar{f}}(\xi,y,z,t) = \frac{\int_0^{T_2} e^{-\eta y}\bar{f}(\xi,y,z,t)dy}{(1-e^{-yT_2})}$$

$$= \frac{1}{(1-e^{-\eta T_2})}\int_0^{T_2} e^{-\eta y}\left\{\frac{\int_0^{T_1} e^{-\xi x} f(x,y,z,t)dx}{(1-e^{-\zeta T_1})}\right\}dy$$

$$= \frac{\int_0^{T_1}\int_0^{T_2} e^{-(\xi x+\eta y)} f(x,y,z,t)dxdy}{(1-e^{-\zeta T_1})(1-e^{-\eta T_2})} \ldots (19)$$

Similarly, for $\gamma \in c_2$ and Re$(\gamma) > |Imj(\gamma)|$ taking the bicomplex Laplace Transform of (19) with respect to z, we have

$$L_z[\bar{\bar{f}}(\xi,\eta,z,t)] = \bar{\bar{\bar{f}}}(\xi,\eta,\gamma,t) = \frac{\int_0^{T_3} e^{-\gamma z}\bar{\bar{f}}(\xi,\eta,z,t)dz}{(1-e^{-\gamma T_3})} =$$
$$\frac{1}{(1-e^{-\gamma T_3})}\int_0^{T_3} e^{-\gamma z}\left\{\frac{\int_0^{T_1}\int_0^{T_2} e^{-(\xi x+\eta y)} f(x,y,z,t)dxdy}{(1-e^{-\zeta T_1})(1-e^{-\eta T_2})}\right\}dz =$$
$$\frac{\int_0^{T_1}\int_0^{T_2}\int_0^{T_3} e^{-(\xi x+\eta y+\gamma z)} f(x,y,z,t)dxdydz}{(1-e^{-\zeta T_1})(1-e^{-\eta T_2})(1-e^{-\gamma T_3})} \ldots (20)$$

and, for $\kappa \in c_2$ and Re$(\kappa) > |Imj(\kappa)|$ taking the bicomplex Laplace Transform of (20) with respect to t, we have

$$L_t\left[\bar{\bar{\bar{f}}}(\xi,\eta,\gamma,t)\right] = \bar{\bar{\bar{\bar{f}}}}(\xi,\eta,\gamma,\kappa) = \frac{\int_0^{T_4} e^{-\kappa t}\bar{\bar{\bar{f}}}(\xi,\eta,\gamma,t)dt}{(1-e^{-\kappa T_4})} =$$
$$\frac{1}{(1-e^{-\kappa T_4})}\int_0^{T_4} e^{-\kappa t}\left\{\frac{\int_0^{T_1}\int_0^{T_2}\int_0^{T_3} e^{-(\xi x+\eta y+\gamma z)} f(x,y,z,t)dxdydz}{(1-e^{-\zeta T_1})(1-e^{-\eta T_2})(1-e^{-\gamma T_3})}\right\}dt =$$
$$\frac{\int_0^{T_1}\int_0^{T_2}\int_0^{T_3}\int_0^{T_4} e^{-(\xi x+\eta y+\gamma z+\kappa t)} f(x,y,z,t)dxdydzdt}{(1-e^{-\zeta T_1})(1-e^{-\eta T_2})(1-e^{-\gamma T_3})(1-e^{-\kappa T_4})} \ldots (21)$$

Thus

$$L_{x,y,z,t}[f(x,y,z,t)] = \frac{\int_0^{T_1}\int_0^{T_2}\int_0^{T_3}\int_0^{T_4} e^{-(\xi x+\eta y+\gamma z+\kappa t)} f(x,y,z,t)dxdydzdt}{(1-e^{-\zeta T_1})(1-e^{-\eta T_2})(1-e^{-\gamma T_3})(1-e^{-\kappa T_4})} = \bar{\bar{\bar{\bar{f}}}}(\xi,\eta,\gamma,\kappa) \ldots (22)$$

[5] CONCLUSION

In this paper we evaluated the fourth order bicomplex Laplace transform of periodic function which is natural extention of complex triple Laplace transform. It is applicable on solving some kind of fourth order differential equation of bicomplex valued function due to large class of frequency domain. Bicomplex numbers being basically four dimensional hypercomplex numbers, provide large class of frequency domain.

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